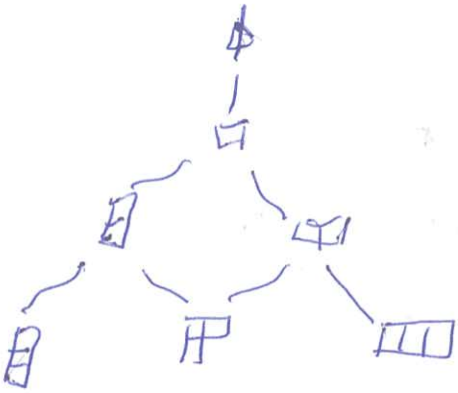
11.10.2025 (1)
Adv. Discrete Math
A. Ram

The Young lattice



$\mathbb{Y} = \bigsqcup_{k\in\mathbb{Z}_{\ge 0}} \mathbb{Y}(k)$ where $\mathbb{Y}(k) = \left\{\begin{matrix}\text{partitions with} \\ k \text{ boxes}\end{matrix}\right\}$

and the partial order is given by inclusion $\mu \subseteq \lambda$.

This describes the tensor product decomposition of

$$B(\mu)\otimes B(\square) = \bigoplus_{\substack{\lambda \vdash k+1 \\ \lambda \supseteq \mu}} B(\lambda)$$

Recall that

$$\left\{\begin{matrix}\text{standard tableaux} \\ \text{of shape } \lambda\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{paths } \emptyset \text{ to } \lambda \\ \text{in } \mathbb{Y}\end{matrix}\right\}$$

Schur-Weyl duality (crystal version)

$$B(\square)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} B(\lambda) \otimes \hat{S}_k^{\lambda}$$

$$\boxed{i_1}\otimes \cdots \otimes \boxed{i_k} \longmapsto (P, Q)$$

where P is a SSYT of shape λ filled from $\{1, \dots, n\}$

Q is a standard tableau of shape λ.

The S_k-crystal

$$\hat{S}_k^\lambda = \{\text{standard tableaux of shape } \lambda\}$$

with $\tilde{s}_1, \dots, \tilde{s}_{k-1}$ given by

$\tilde{s}_i Q$ is Q with i and $i+1$ switched if the result is standard and $\tilde{s}_i Q = 0$ otherwise.

$$\begin{tikzcd}
 & \begin{array}{ll} 1 & 2 \\ 3 & 4 \\ 5 & \end{array} \arrow[dl, dash] & \text{row reading tableau} \\
\begin{array}{ll} 1 & 3 \\ 2 & 4 \\ 5 & \end{array} \arrow[dr] & & \begin{array}{ll} 1 & 2 \\ 3 & 5 \\ 4 & \end{array} \arrow[ul] \arrow[dl] \\
 & \begin{array}{ll} 1 & 3 \\ 2 & 5 \\ 4 & \end{array} \arrow[d, dash] & \\
 & \begin{array}{ll} 1 & 4 \\ 2 & 5 \\ 3 & \end{array} & \text{column reading tableau}
\end{tikzcd}$$

$(\hat{S}_n^\lambda, \tilde{s}_1, \dots, \tilde{s}_{k-1})$ is an S_k-crystal.

$$B(\square)^{\otimes k} \cong \bigoplus_{\lambda \vdash k} B(\lambda) \otimes \hat{S}_k^\lambda$$

is a $GL_n \times S_k$ crystal.

12.10.2025
Adv. Disc. Math (3)

A. Ram

Knuth equivalence If $x, y, z \in \{\boxed{1}, \ldots, \boxed{n}\}$ define

(a) If $x \le y < z$ then $xzy \overset{3}{\longleftrightarrow} zxy$

(b) If $x < y \le z$ then $yzx \overset{3}{\longleftrightarrow} yxz$

Show that Knuth equivalence describes the orbits of $\tilde{S}_k$ in $B(\square)^{\otimes k}$.

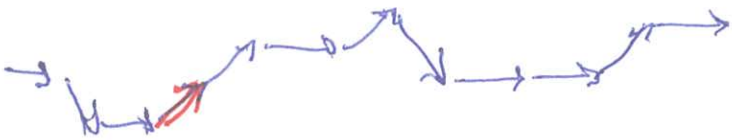
Describing $\tilde{f}_j$ on $B(\square)^{\otimes k}$

Let
$$\boxed{i_1} \otimes \cdots \otimes \boxed{i_k} \in B(\square)^{\otimes k}$$
Convert the word to a path by
$$\boxed{j} = \nearrow \qquad \boxed{j+1} = \searrow \quad \text{and} \quad \boxed{r} = \longrightarrow \text{ for } r \notin \{j, j+1\}$$
Let ℓ be the last $\nearrow$ step from the minimum height. Then
$$\tilde{f}_j(\boxed{i_1} \otimes \cdots \otimes \boxed{i_k}) =$$ is the same except with $\boxed{i_\ell}$ changed from $\boxed{j}$ to $\boxed{j+1}$

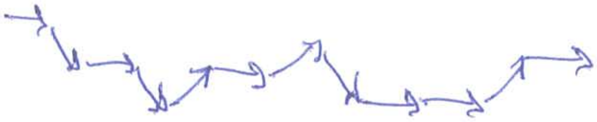
Examples: $\tilde{f}_3(1\,3\,4\,2\,2\,1\,2\,3\,4\,1\,2\,4)$

$= \cancel{134221234123}\ .\ 0$

$\tilde{f}_2(1\ 3\ 4\ 2\ 2\ 1\ 2\ 3\ 4\ 1\ 2\ 4)$



$= 1\ 3\ 4\ 3\ 2\ 1\ 2\ 3\ 4\ 1\ 2\ 4$



11.10.2023

Adv. Disc. Math

A. Ram

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$B(\square)\otimes B(\square)\otimes B(\square)$

$\left(\boxed{1}\boxed{1}\boxed{1}\,,\ \boxed{1}\boxed{2}\boxed{3}\right)$

$$\begin{tikzcd}
111 \arrow[r, "\tilde f_1"] & 211 \arrow[r, "\tilde f_1"] \arrow[d, "\tilde f_2"] & 221 \arrow[r, "\tilde f_1"] \arrow[d, "\tilde f_2"] & 222 \arrow[d, "\tilde f_2"] \\
 & 311 \arrow[r, "\tilde f_1"] & 321 \arrow[r, "\tilde f_1"] \arrow[d, "\tilde f_2"] & 322 \arrow[d, "\tilde f_2"] \\
 & & 331 \arrow[r, "\tilde f_1"] & 332 \arrow[d, "\tilde f_2"] \\
 & & & 333
\end{tikzcd}$$

$\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right)$

$$\begin{tikzcd}
121 \arrow[r, "1"] \arrow[d, "2"] & 122 \arrow[r, "2"] & 132 \arrow[r, "2"] & 133 \arrow[d, "1"] \\
131 \arrow[r, "1"] & 231 \arrow[r, "1"] & 232 \arrow[r, "2"] & 233
\end{tikzcd}$$

$\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right)$

$$\begin{tikzcd}
112 \arrow[r, "1"] \arrow[d, "2"] & 212 \arrow[r, "2"] & 312 \arrow[r, "2"] & 313 \arrow[d, "1"] \\
113 \arrow[r, "1"] & 213 \arrow[r, "1"] & 223 \arrow[r, "2"] & 323
\end{tikzcd}$$

123

$\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right)$