Partitions and the Young lattice.

A <u>partition</u> is $(\lambda_1, \lambda_2, \ldots \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

Identify λ with a set of boxes

$$\lambda = \{ (r,c) \mid r \in \{1,\ldots,n\} \text{ and } c \in \{1,\ldots,\lambda_r\} \}$$

A <u>standard tableau</u> of shape λ is a function

$$T : \lambda \to \{1,\ldots,k\}, \text{ where } k = \lambda_1 + \cdots + \lambda_n$$

such that

(a) If $(r,c), (r,c+1) \in \lambda$ then $T(r,c) < T(r,c+1)$

(b) If $(r,c), (r+1,c) \in \lambda$ then $T(r,c) < T(r+1,c)$

A <u>semistand Young tableau</u> <u>of shape λ filled from</u> $\{1,\ldots,n\}$ is a function $P : \lambda \to \{1,\ldots,n\}$ such that

(a) If $(r,c), (r,c+1) \in \lambda$ then $P(r,c) \leq P(r,c+1)$

(b) If $(r,c), (r,c+1) \in \lambda$ then $P(r,c) < P(r+1,c)$

<u>Example</u> $n=3$ and $\lambda = (2,1)$

$\begin{array}{ll} 1 & 1 \\ 2 & \end{array}$

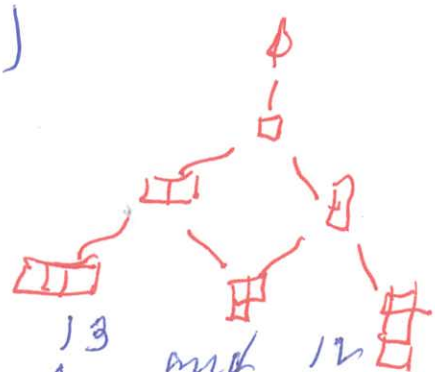
$\begin{array}{ll} 1 & 2 \\ 2 & \end{array}$ $\quad$ $\begin{array}{ll} 1 & 1 \\ 3 & \end{array}$

$\begin{array}{ll} 1 & 3 \\ 2 & \end{array}$ $\quad$ $\begin{array}{ll} 1 & 2 \\ 3 & \end{array}$

$\begin{array}{ll} 2 & 2 \\ 3 & \end{array}$ $\quad$ $\begin{array}{ll} 1 & 3 \\ 3 & \end{array}$

$\begin{array}{ll} 2 & 3 \\ 3 & \end{array}$

SSYT$_3$



$\begin{array}{ll} 1 & 3 \\ 2 & \end{array}$ and $\begin{array}{ll} 1 & 2 \\ 3 & \end{array}$

standard tableaux.

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Crystals are obtained from

$$B(\square) = \left(\boxed{1} \xrightarrow{\tilde{f}_1} \boxed{2} \xrightarrow{\tilde{f}_2} \cdots \xrightarrow{\tilde{f}_{n-1}} \boxed{n} \right)$$

$$\varepsilon_1 \qquad \varepsilon_2 \qquad \varepsilon_n$$

by taking tensor products. Irreducible crystals have a unique highest weight element $b \in B$ such that if $i \in \{1, \ldots, n-1\}$ then

$$\tilde{e}_i b = 0.$$

A <u>word of length k in the alphabet $\{1, \ldots, n\}$</u> is an element of

$$B(\square)^{\otimes k} = \{ b_{i_1} \otimes \cdots \otimes b_{i_k} \mid i_1, \ldots, i_k \in \{1, \ldots, n\} \}$$

The generating function of words of length k in n letters is

$$\mathrm{char}(B(\square)^{\otimes k}) = (x_1 + \cdots + x_n)^k = p_1(x_1, \ldots, x_n)^k.$$

<u>Theorem</u> Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with k boxes. Let

$$B(\lambda) = \{ \text{SSYTs of shape } \lambda \text{ filled from } \{1, \ldots, n\} \}$$

There is a unique crystal structure on $B(\lambda)$ such that

$$B(\lambda) \longrightarrow B(\square)^{\otimes k}$$
$$P \longmapsto (\text{arabic reading of } P)$$

is a crystal isomorphism.

Theorem Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with k boxes. Let Q be a standard tableau of shape λ.

$$b_Q = \boxed{r(Q^{-1}(1))} \otimes \cdots \otimes \boxed{r(Q^{-1}(k))}$$

is a highest weight element of $B(\square)^{\otimes k}$ and $\mathrm{wt}(b_Q) = \lambda$.

Here $r(Q^{-1}(j))$ is the row index of the box containing j in Q.

Let $n=3$:

$B(\square)$ $\qquad 1 \xrightarrow{\tilde f_1} 2 \xrightarrow{\tilde f_2} 3$

$B(\square) \otimes B(\square)$

$$\begin{tikzcd}
11 \arrow[d, "\tilde f_1"'] & 12 \arrow[r, "\tilde f_2"] & 13 \arrow[d, "\tilde f_1"] \\
21 \arrow[r, "\tilde f_1"] \arrow[d, "\tilde f_2"'] & 22 \arrow[d, "\tilde f_2"] & 23 \\
31 \arrow[r, "\tilde f_1"'] & 32 \arrow[r, "\tilde f_2"'] & 33
\end{tikzcd}$$

$Q = \boxed{1\,|\,2} \longmapsto 11$

$Q = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \longmapsto 12$

$B(\begin{smallmatrix}\square\\\square\end{smallmatrix})$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \xrightarrow{\tilde f_2} \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \xrightarrow{\tilde f_1} \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$$

$B(\square\square)$

$$\begin{tikzcd}
\boxed{1\,|\,1} \arrow[r, "\tilde f_1"] & \boxed{1\,|\,2} \arrow[r, "\tilde f_2"] \arrow[d, "\tilde f_1"'] & \boxed{1\,|\,3} \arrow[d, "\tilde f_1"] \\
 & \boxed{2\,|\,2} \arrow[r, "\tilde f_2"'] & \boxed{2\,|\,3} \arrow[d, "\tilde f_2"] \\
 & & \boxed{3\,|\,3}
\end{tikzcd}$$

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$B(\square)\otimes B(\square)\otimes B(\square)$

$\left(\boxed{1\,|\,1\,|\,1}\,,\ \boxed{1\,|\,2\,|\,3} \right)$

$$\begin{tikzcd}
111 \arrow[r, "\tilde f_1"] & 211 \arrow[r, "\tilde f_1"] \arrow[d, "\tilde f_2"'] & 221 \arrow[r, "\tilde f_1"] \arrow[d, "\tilde f_2"'] & 222 \arrow[d, "\tilde f_2"'] \\
 & 311 \arrow[r, "\tilde f_1"] & 321 \arrow[r, "\tilde f_1"] \arrow[d, "\tilde f_2"'] & 322 \arrow[d, "\tilde f_2"'] \\
 & & 331 \arrow[r, "\tilde f_1"] & 332 \arrow[d, "\tilde f_2"'] \\
 & & & 333
\end{tikzcd}$$

$\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right)$

$$\begin{tikzcd}
121 \arrow[r, "1"] \arrow[d, "2"] & 122 \arrow[r, "2"] & 132 \arrow[r, "2"] & 133 \arrow[d, "1"] \\
131 \arrow[r, "1"] & 231 \arrow[r, "1"] & 232 \arrow[r, "2"] & 233
\end{tikzcd}$$

$\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,,\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \right)$

$$\begin{tikzcd}
112 \arrow[r, "1"] \arrow[d, "2"] & 212 \arrow[r, "2"] & 312 \arrow[r, "2"] & 313 \arrow[d, "1"] \\
113 \arrow[r, "1"] & 213 \arrow[r, "1"] & 223 \arrow[r, "2"] & 323
\end{tikzcd}$$

123

$\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \right)$