09.09.2025 (1)
Adv. Discr. Math
A. Ram

Crystals Let $n \in \mathbb{Z}_{>0}$.

$$B(\square) = \left(\boxed{1} \xrightarrow{\tilde f_1} \boxed{2} \xrightarrow{\tilde f_2} \cdots \xrightarrow{\tilde f_{n-1}} \boxed{n}\right)$$

Given crystals
$$B_1 = (B_1, wt^{(1)}, \tilde f_1^{(1)}, \dots, \tilde f_{n-1}^{(1)})$$
$$B_2 = (B_2, wt^{(2)}, \tilde f_1^{(2)}, \dots, \tilde f_{n-1}^{(2)})$$
the tensor product is $B_1 \otimes B_2 = (B_1 \times B_2, wt, \tilde f_1, \dots, \tilde f_{n-1})$ given by
$$wt(b_1 \otimes b_2) = wt^{(1)}(b_1) + wt^{(2)}(b_2) \quad \text{and}$$
$$\tilde f_i(b_1 \otimes b_2) = \begin{cases} \tilde f_i^{(1)} b_1 \otimes b_2, & \text{if } d_i^+(b_1) > d_i^-(b_2), \\ b_1 \otimes \tilde f_i b_2, & \text{if } d_i^+(b_1) \le d_i^-(b_2). \end{cases}$$

Example $n=3$: $B(\square) = \left(\boxed{1} \xrightarrow{\tilde f_1} \boxed{2} \xrightarrow{\tilde f_2} \boxed{3}\right)$

$B(\square) \otimes B(\square) =$
$$\begin{tikzcd}
11 \arrow[r] \arrow[d, "\tilde f_1"'] & 12 \arrow[r, "\tilde f_2"] & 13 \arrow[d, "\tilde f_1"] \\
21 \arrow[r, "\tilde f_1"] \arrow[d, "\tilde f_2"'] & 22 \arrow[d, "\tilde f_2"] & 23 \\
31 \arrow[r, "\tilde f_1"] & 32 \arrow[r, "\tilde f_2"'] & 33
\end{tikzcd}$$

A **morphism** from B_1 to B_2 is a function $\Phi: B_1 \to B_2$ such that
$$wt^{(2)}(\Phi(b_1)) = wt^{(1)}(b_1) \quad \text{and}$$
$$\Phi(\tilde f_i^{(1)} b_1) = \tilde f_i^{(2)} \Phi(b_2)$$

04.04.2023
Adv. Disc Math (2)

A subcrystal of B is a subset of B closed A. Ram
under the operators $\tilde{e}_i$ and $\tilde{f}_i$.

A crystal is irreducible, or simple, if B
has no subcrystals except ϕ and B.

A highest weight element in a crystal B
is $b \in B$ such that

if $i \in \{1, \dots, n-1\}$ then $\tilde{e}_i b = 0$.

Let

$B^+ = \left\{ \begin{matrix} \text{highest weight} \\ \text{elements of } B \end{matrix} \right\}$ and $B^+_\lambda = \{ b \in B^+ \mid wt(b) = \lambda \}$

Theorem

(a) A crystal B is irreducible if and only if the graph of B is connected.

(b) A crystal is irreducible if and only if $Card(B^+) = 1$.

Proposition Assume B_1 and B_2 are irreducible crystals

(a) If $\Phi : B_1 \to B_2$ is a crystal morphism then Φ is a crystal isomorphism and $B_1 \simeq B_2$

(b) If $B_1^+ = \{b_1^+\}$ and $B_2^+ = \{b_2^+\}$ then

$B_1 \simeq B_2$ if and only if $wt(b_1^+) = wt(b_2^+)$

The character of a crystal B is

$$\mathrm{char}(B) = \sum_{p\in B} x^{wt(p)} \quad \text{where } x^{\mu} = x_1^{\mu_1}\cdots x_n^{\mu_n}$$

if $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$

<u>Theorem</u> Let B_1 and B_2 be crystals

$$\mathrm{char}(B_1 \oplus B_2) = \mathrm{char}(B_1) + \mathrm{char}(B_2)$$

$$\mathrm{char}(B_1 \otimes B_2) = \mathrm{char}(B_1)\,\mathrm{char}(B_2)$$

and

$B_1 \cong B_2$ if and only if $\mathrm{char}(B_1) = \mathrm{char}(B_2)$.

<u>Examples</u> $\mathrm{char}(B(\square)) = x_1 + x_2 + x_3 = p_1$

$\mathrm{char}(B_1) = x_1^2 + x_1x_2 + x_1x_3 + x_2x_3 + x_2^2 + x_3^2 = h_2$

$\mathrm{char}(B_2) = x_1x_2 + x_1x_3 + x_2x_3 = e_2$

$\mathrm{char}(B(\square)\otimes B(\square)) = p_1^2 = e_2 + h_2$

$$\begin{tikzcd}
111 \arrow[r, "1"] & 211 \arrow[r, "1"] \arrow[d, "2"] & 221 \arrow[r, "1"] \arrow[d, "2"] & 222 \arrow[d, "2"] \\
& 311 \arrow[r, "1"] & 321 \arrow[r, "1"] \arrow[d, "2"] & 322 \arrow[d, "2"] \\
& & 331 \arrow[r, "1"] & 332 \arrow[d, "2"] \\
& & & 333
\end{tikzcd}$$

$$\begin{tikzcd}
121 \arrow[r, "1"] \arrow[d, "2"] & 122 \arrow[r, "2"] & 132 \arrow[r, "2"] & 133 \arrow[d, "1"] \\
131 \arrow[r, "1"] & 231 \arrow[r, "1"] & 232 \arrow[r, "2"] & 233
\end{tikzcd}$$

$$\begin{tikzcd}
112 \arrow[r, "1"] \arrow[d, "2"] & 212 \arrow[r, "2"] & 312 \arrow[r, "2"] & 313 \arrow[d, "1"] \\
113 \arrow[r, "1"] & 213 \arrow[r, "2"] & 223 \arrow[r, "2"] & 323
\end{tikzcd}$$

123

$$B(\square)^{\otimes 3} = B(\square) \otimes B(\square) \otimes B(\square)$$

$$\cong \left(B(\yng(2)) \oplus B(\yng(1,1))\right) \otimes B(\square)$$

$$\cong \left(B(\yng(2)) \otimes B(\square)\right) \oplus \left(B(\yng(1,1)) \otimes B(\square)\right)$$

$$\cong B(\yng(3)) \oplus B(\yng(2,1)) \oplus B(\yng(2,1)) \oplus B(\yng(1,1,1))$$