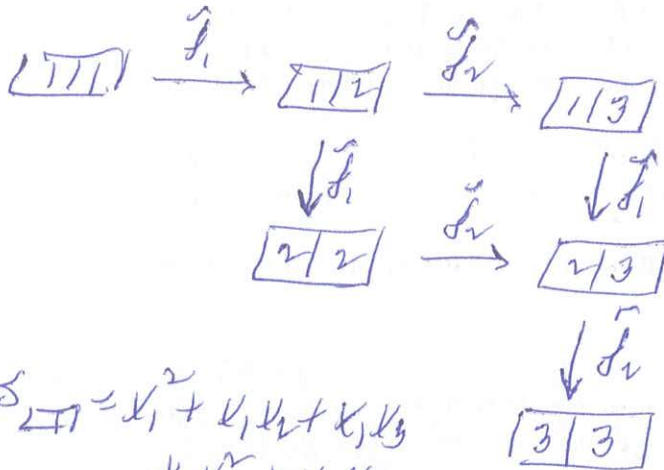
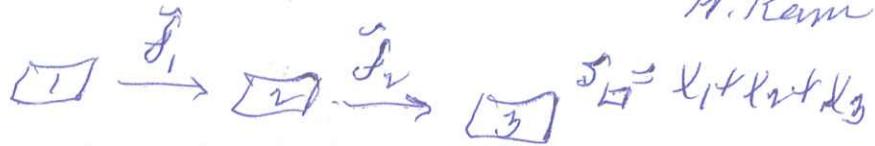


Crystals

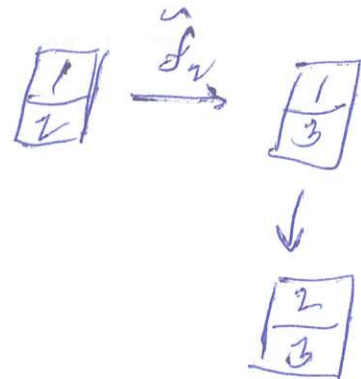
05.09.2025
Adv Dis Math
A. Rem

①

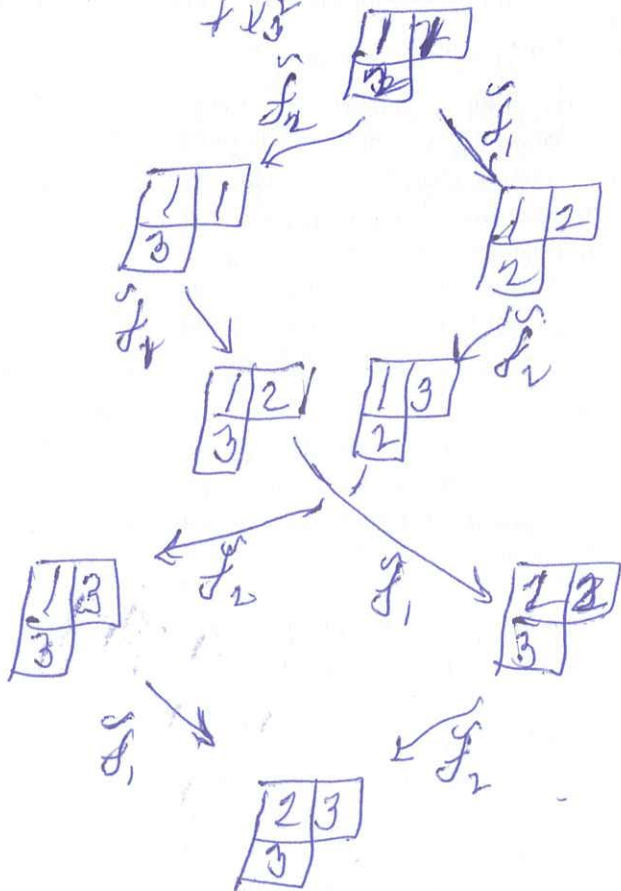
Examples:



$$S_{\square} = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$



$$S_{\square} = x_1 x_2 + x_1 x_3 + x_2 x_3$$



$$S_{\square} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

Let $n \in \mathbb{Z}_{>0}$

A crystal is an object in the category ^{Math} of crystals.

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(2)

The category of crystals is the subcategory of the category of WB-crystals generated by $B(\square)$ (under tensor product).

$B(\square)$ is the crystal

$$\boxed{1} \xrightarrow{\hat{f}_1} \boxed{2} \xrightarrow{\hat{f}_2} \dots \xrightarrow{\hat{f}_{n-1}} \boxed{n}$$

with $\text{wt}(\boxed{i}) = \varepsilon_i$.

Roots and simple roots

Define $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{Z}^n$ by

$$\varepsilon_i = (0, \dots, 0, \overset{i\text{th}}{1}, 0, \dots, 0)^t$$

The set of roots is

$$R = \{ \varepsilon_i - \varepsilon_j \mid i, j \in \{1, \dots, n\} \text{ and } i \neq j \}$$

The simple roots are $\alpha_1, \dots, \alpha_{n-1}$ and given by

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \text{ for } i \in \{1, \dots, n-1\}.$$

Frameworks and WB crystals

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A. Ram

A framework is a set B with a function

$$wt: B \rightarrow \mathbb{Z}^n$$

A WB-crystal is a framework (B, wt) with

$$\tilde{f}_1, \dots, \tilde{f}_{n-1} \in M_{B \times B}(\{0, 1\})$$

such that

(a) $\tilde{f}_i$ has at most one 1 in each column

(b) If $(\tilde{f}_i)_{b'b} = 1$ then $wt(b') = wt(b) - \alpha_i$

Write

$$\tilde{f}_i b = \begin{cases} b', & \text{if } (\tilde{f}_i)_{b'b} = 1 \\ 0 & \text{if all entries in col. } b \text{ are } 0 \end{cases}$$

i-strings

Let $(B, wt, \tilde{f}_1, \dots, \tilde{f}_{n-1})$ be a WB-crystal.

Define $\tilde{e}_1, \dots, \tilde{e}_{n-1}$ by

$$\tilde{e}_i = \tilde{f}_i^+$$

For $b \in B$ define $d_i^+(b), d_i^-(b) \in \mathbb{Z}_{\geq 0}$ by

$$\tilde{e}_i^{d_i^+(b)} b \neq 0 \text{ and } \tilde{e}_i^{d_i^+(b)+1} b = 0 \text{ and}$$

$$\tilde{f}_i^{d_i^-(b)} b \neq 0 \text{ and } \tilde{f}_i^{d_i^-(b)+1} b = 0.$$

The i -string of b is

$$S_i(b) = \{ \tilde{f}_i^{\tilde{d}_i^+(b)} b, \dots, \tilde{f}_i^{\tilde{d}_i^-(b)} b, \tilde{f}_i b, b, \tilde{e}_i b, \tilde{e}_i^{\tilde{d}_i^-(b)} b, \dots, \tilde{e}_i^{\tilde{d}_i^+(b)} b \}$$

Tensor products and direct sums

$$\text{Let } (B_1, \text{wt}^{(1)}, \tilde{f}_1^{(1)}, \dots, \tilde{f}_{n-1}^{(1)})$$

$$(B_2, \text{wt}^{(2)}, \tilde{f}_1^{(2)}, \dots, \tilde{f}_{n-1}^{(2)})$$

be WB-crystals. The tensor product is

$$B_1 \otimes B_2 = (B_1 \times B_2, \text{wt}, \tilde{f}_1, \dots, \tilde{f}_{n-1})$$

given by

$$\text{wt}(b_1 \otimes b_2) = \text{wt}^{(1)}(b_1) + \text{wt}^{(2)}(b_2)$$

(write $b_1 \otimes b_2$ for (b_1, b_2) in $B_1 \times B_2$) and

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i^{(1)} b_1 \otimes b_2, & \text{if } d_i^+(b_1) > d_i^-(b_2), \\ b_1 \otimes \tilde{f}_i^{(2)} b_2, & \text{if } d_i^+(b_1) \leq d_i^-(b_2). \end{cases}$$