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Crystals

Examples:

$$\boxed{1} \xrightarrow{\tilde{f}_1} \boxed{2} \xrightarrow{\tilde{f}_2} \boxed{3} \qquad s_{\square} = x_1+x_2+x_3$$

$$\begin{tikzcd}
\boxed{1\,|\,1} \arrow[r, "\tilde{f}_1"] & \boxed{1\,|\,2} \arrow[r, "\tilde{f}_2"] \arrow[d, "\tilde{f}_1"] & \boxed{1\,|\,3} \arrow[d, "\tilde{f}_1"] \\
 & \boxed{2\,|\,2} \arrow[r, "\tilde{f}_2"] & \boxed{2\,|\,3} \arrow[d, "\tilde{f}_2"] \\
 & & \boxed{3\,|\,3}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \arrow[r, "\tilde{f}_2"] & \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \arrow[d] \\
 & \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}
\end{tikzcd}$$

$$s_{\square\square} = x_1^2+x_1x_2+x_1x_3 + x_2^2+x_2x_3 + x_3^2$$

$$s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = x_1x_2+x_1x_3+x_2x_3$$

$$\begin{tikzcd}
 & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \arrow[dl, "\tilde{f}_2"'] \arrow[dr, "\tilde{f}_1"] & \\
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \arrow[d, "\tilde{f}_1"'] & & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \arrow[d, "\tilde{f}_2"] \\
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \arrow[d, "\tilde{f}_1"'] & & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \arrow[d, "\tilde{f}_2"] \\
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \arrow[dr, "\tilde{f}_2"'] & & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \arrow[dl, "\tilde{f}_1"] \\
 & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} &
\end{tikzcd}$$

$$s_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

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Let $n \in \mathbb{Z}_{>0}$

A <u>crystal</u> is an object in the category of crystals.

The <u>category of crystals</u> is the subcategory of the category of WB-crystals generated by $B(\square)$ (under tensor product).

$B(\square)$ is the crystal

$$\boxed{1} \xrightarrow{\tilde{f}_1} \boxed{2} \xrightarrow{\tilde{f}_2} \cdots \xrightarrow{\tilde{f}_{n-1}} \boxed{n}$$

with $wt(\boxed{i}) = \varepsilon_i$.

<u>Roots and simple roots</u>

Define $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{Z}^n$ by

$$\varepsilon_i = (0, \ldots, 0, \overset{i\text{th}}{1}, 0, \ldots 0)^t$$

The <u>set of roots</u> is

$$R = \{\varepsilon_i - \varepsilon_j \mid i,j \in \{1, \ldots, n\} \text{ and } i \neq j\}.$$

The <u>simple roots</u> are $\alpha_1, \ldots, \alpha_{n-1}$ given by

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad \text{for } i \in \{1, \ldots, n-1\}.$$

Frameworks and WB crystals

A framework is a set B with a function

$$\mathrm{wt}: B \longrightarrow \mathbb{Z}^n$$

A WB-crystal is a framework (B, wt) with

$$\tilde{f}_1, \ldots, \tilde{f}_{n-1} \in M_{B\times B}(\mathbb{F}_0, \mathbb{R})$$

such that

(a) $\tilde{f}_i$ has at most one 1 in each column

(b) If $(\tilde{f}_i)_{b'b} = 1$ then $\mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i$

Write

$$\tilde{f}_i b = \begin{cases} b', & \text{if } (\tilde{f}_i)_{b'b} = 1 \\ 0 & \text{if all entries in col. } b \text{ are } 0 \end{cases}$$

i-strings

Let $(B, \mathrm{wt}, \tilde{f}_1, \ldots, \tilde{f}_{n-1})$ be a WB-crystal.

Define $\tilde{e}_1, \ldots, \tilde{e}_{n-1}$ by

$$\tilde{e}_i = \tilde{f}_i^{\,t}$$

For $b \in B$ define $d_i^+(b), d_i^-(b) \in \mathbb{Z}_{\geq 0}$ by

$\tilde{e}_i^{\,d_i^+(b)} b \neq 0$ and $\tilde{e}_i^{\,d_i^+(b)+1} = 0$ and

$\tilde{f}_i^{\,d_i^-(b)} b \neq 0$ and $\tilde{f}_i^{\,d_i^-(b)+1} = 0$.

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The i-string of b is

$$S_i(b) = \{ \tilde{f}_i^{\,d_i^-(b)}, \ldots, \tilde{f}_i^2 b, \tilde{f}_i b, b, \tilde{e}_i b, \tilde{e}_i^2 b, \ldots, \tilde{e}_i^{\,d_i^+(b)} b \}.$$

Tensor products and direct sums

Let
$$(B_1, wt^{(1)}, \tilde{f}_1^{(1)}, \ldots, \tilde{f}_{n-1}^{(1)})$$
$$(B_2, wt^{(2)}, \tilde{f}_1^{(2)}, \ldots, \tilde{f}_{n-1}^{(2)})$$
be WB-crystals. The tensor product is
$$B_1 \otimes B_2 = (B_1 \times B_2, wt, \tilde{f}_1, \ldots, \tilde{f}_{n-1})$$
given by
$$wt(b_1 \otimes b_2) = wt^{(1)}(b_1) + wt^{(2)}(b_2)$$
(write $b_1 \otimes b_2$ for (b_1, b_2) in $B_1 \times B_2$) and

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i^{(1)} b_1 \otimes b_2, & \text{if } d_i^+(b_1) > d_i^-(b_2), \\ b_1 \otimes \tilde{f}_i^{(2)} b_2, & \text{if } d_i^+(b_1) \le d_i^-(b_2). \end{cases}$$