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Binomial theorems

Positive version If $n\in\mathbb{Z}_{>0}$ then

$$(u+z)^n = \sum_{r=0}^{n} u^{n-r}z^r\binom{n}{r}$$

Negative version If $n\in\mathbb{Z}_{>0}$ then

$$(u+z)^{-n} = \sum_{r\in\mathbb{Z}} u^{-n-r}z^r\binom{n+r-1}{r}$$

These are the cases $x_1=x_2=\cdots=x_n=1$ of

$$\prod_{i=1}^{n}(u+x_iz) = u^n\prod_{i=1}^{n}(1+zu^{-1}x_i) = \sum_{r=0}^{n}u^{n-r}z^r e_r(x)$$

$$\prod_{i=1}^{n}\frac{1}{u-x_iz} = u^{-n}\prod_{i=1}^{n}\frac{1}{1-zu^{-1}x_i} = \sum_{r\in\mathbb{Z}_{\geq 0}} z^r u^{-n-r}h_r(x)$$

where

$$e_r = \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} x_{i_1}\cdots x_{i_r} \quad \text{and}$$

$$h_r = \sum_{1\leq i_1\leq\cdots\leq i_r\leq n} x_{i_1}\cdots x_{i_r}$$

so that

$$e_r(1,1,\ldots,1) = \binom{n}{r}$$

$$h_r(1,1,\ldots,1) = \binom{n+r-1}{r}$$

Finite
q-binomial theorems

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positive version If $n \in \mathbb{Z}_{>0}$ then

$$\prod_{i=1}^{n}(u+q^{i-1}z) = \sum_{r=0}^{n} q^{\frac{1}{2}r(r-1)} \begin{bmatrix} n \\ r \end{bmatrix} u^{n-r} z^r$$

negative version If $n \in \mathbb{Z}_{>0}$ then

$$\prod_{i=1}^{n} \frac{1}{(u-q^{i-1}z)} = \sum_{r \in \mathbb{Z}_{\geq 0}} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} u^{n-r} z^r.$$

These are (a) and (b) where

$$x_1=1,\ x_2=q,\ x_3=q^2, \ldots, x_n = q^{n-1}$$

In other words:

$$e_r(1, q, q^2, \ldots, q^{n-1}) = \begin{bmatrix} n \\ r \end{bmatrix}$$

$$h_r(1, q, q^2, \ldots, q^{n-1}) = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}.$$

The infinite q-binomial theorem (general version).

$$\frac{(tz;q)_\infty}{(uz;q)_\infty} = \prod_{i=1}^{\infty} \frac{(1-tq^{i-1};z)}{(1-uq^{i-1};z)}$$

$$= \sum_{r \in \mathbb{Z}_{\geq 0}} u^r \frac{(tu^{-1};q)_r}{(q;q)_r} z^r.$$

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If $t = q^n$ and $u = 1$ then the left hand side is

$$\frac{(q^n z;q)_\infty}{(z;q)_\infty} = \frac{1}{(z;q)_n} = \prod_{i=1}^{n} \frac{1}{1-q^{i-1}z}$$

to give the finite q-binomial theorem (neg. version).

<u>Proof of the general infinite q-binomial theorem</u>

Recognize that

$$L(z;q,t,u) = \frac{(tz;q)_\infty}{(uz;q)_\infty} \text{ satisfies}$$

$$L(z;q,t,u) = \frac{(1-tz)}{(1-uz)} L(qz;q,t,u).$$

which provides a recursion on the coefficients of

$$L(z;q,t,u) = \sum_{r \in \mathbb{Z}_{\geq 0}} c_r(q,t,u) z^r \quad \text{as}$$

$$c_r(q,t,u)q^r - t c_{r-1}(q,t,u) q^{r-1} = c_r(q,t,u) - u c_{r-1}(q,t,u)$$

so that

$$c_r(q,t,u) = c_{r-1}(q,t,u) \frac{u - tq^{r-1}}{1-q\cdot q^{r-1}} = u^r \frac{(tu^{-1};q)_r}{(q;q)_r}.$$