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Adv. Disc. Math

Power sums and the Cauchy-Macdonald kernel

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The power sums $p_r \in \mathbb{C}[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ are

$p_0 = 1$ and $p_r = x_1^r + \cdots + x_n^r$ for $r \in \mathbb{Z}_{>0}$.

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^n_{\geq 0}$ define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}.$$

Since

$$\log(1-z) = \int \frac{-1}{1-z}\, dz$$

$$= \int -(1+z+z^2+\cdots)\, dz$$

$$= -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 + \cdots = \sum_{r\in\mathbb{Z}_{>0}} \frac{1}{r} z^r$$

then

$$\log\left(\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty}\right) = \sum_{i=1}^n \sum_{\ell\in\mathbb{Z}_{\geq 0}} \log(1-tx_iq^\ell) - \log(1-ux_iq^\ell)$$

$$= \sum_{i=1}^n \sum_{\ell\in\mathbb{Z}_{\geq 0}} \sum_{r\in\mathbb{Z}_{>0}} \left(\frac{-1}{r} t^r x_i^r q^{\ell r} z^r + \frac{1}{r} u^r x_i^r q^{\ell r} z^r\right)$$

$$= \sum_{\ell\in\mathbb{Z}_{\geq 0}} \sum_{r\in\mathbb{Z}_{>0}} \frac{1}{r}(u^r - t^r) q^{\ell r} p_r z^r$$

$$= \sum_{r\in\mathbb{Z}_{>0}} \left(\frac{u^r - t^r}{1-q^r}\right) \frac{p_r}{r} z^r.$$

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(2)

Define

$$z_\lambda(q,t,u) = z_\lambda \prod_{i=1}^{n} \frac{1-q^{\lambda_i}}{u^{\lambda_i}-t^{\lambda_i}}, \quad \text{where}$$

$$z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots \quad \text{with } m_i = \#\{j \mid \lambda_j = i\}.$$

Taking the exponential of both sides of (*) gives

$$\prod_{i=1}^{n} \frac{(tx_i;z;q)_\infty}{(ux_i;z;q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} \left(\sum_{|\lambda|=r} \frac{1}{z_\lambda(q,t,u)} p_\lambda(x) \right) z^r.$$

So

$$\tilde{g}_r = \sum_{|\lambda|=r} \frac{1}{z_\lambda(q,t,u)} p_\lambda(x) = \sum_{|\lambda|=r} \left(\prod_{i=1}^{\ell(\lambda)} \frac{u^{\lambda_i}-t^{\lambda_i}}{1-q^{\lambda_i}} \right) \frac{p_\lambda}{z_\lambda}$$

<u>The Cauchy-Macdonald kernel</u>

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^n_{\geq 0}$ define

$$\tilde{g}_\lambda = \tilde{g}_{\lambda_1} \tilde{g}_{\lambda_2} \cdots \tilde{g}_{\lambda_\ell} \quad \text{and} \quad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}.$$

Then

$$\prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(ux_iy_j;q)_\infty} = \sum_\lambda \tilde{g}_\lambda(x;q,t,u)\, m_\lambda(y)$$

$$= \sum_\lambda \frac{1}{z_\lambda(q,t,u)} p_\lambda(x) p_\lambda(y).$$