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Fundamental symmetric functions

Define

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) \quad \text{and}$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$$

Define

$$g_r = g_r(x;q,t)$$
$$q_r = q_r(x;q,t)$$
$$h_r = h_r(x)$$
$$e_r = e_r(x)$$

$$\left(\begin{aligned} g_r &= \frac{(t;q)_r}{(q;q)_r} P_{(r)}(x;q,t) \\ q_r &= (1-t)P_{(r)}(x;0,t) \\ h_r &= s_{(r)}(x) \\ e_r &= s_{(1^r)}(x) = P_{(1^r)}(x;0,t) = P_{(1^r)}(x;q,t) \end{aligned} \right)$$

by the generating functions

$$\sum_{r\in\mathbb{Z}_{\geq 0}} g_r z^r = \prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty},$$

$$\sum_{r\in\mathbb{Z}_{\geq 0}} q_r z^r = \prod_{i=1}^n \frac{1-tx_iz}{1-x_iz},$$

$$\sum_{r\in\mathbb{Z}_{\geq 0}} h_r z^r = \prod_{i=1}^n \frac{1}{1-x_iz},$$

$$\sum_{r\in\mathbb{Z}_{\geq 0}} e_r z^r = \prod_{i=1}^r (1+x_iz)$$

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"Extensions"

$\hat{g}_r = \hat{g}_r(x;q,t,u)$ and $\hat{q}_r = \hat{q}_r(x;t,u)$ given by

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \hat{g}_r z^r = \prod_{i=1}^{n} \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty}$$

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \hat{q}_r z^r = \prod_{i=1}^{n} \frac{(1-tx_iz)}{(1-ux_iz)}$$

then

$$\hat{q}_r(x;t,u) = \hat{g}_r(x;0,t,u)$$
$$h_r(x) = \hat{g}_r(x;0,0,1)$$
$$q_r(x;t) = \hat{g}_r(x;0,t,1)$$
$$e_r(x) = \hat{g}_r(x;0,-1,0)$$

Since

$$g_r(x;q,t) = \hat{g}_r(x;q,t,1) \quad \text{and}$$

$$\hat{g}_r(x_1,\ldots,x_n;q,t,u) = u^r g_r(u^{-1}x_1,\ldots,u^{-1}x_n;q,t,u)$$
$$= u^r g_r(x;q,tu^{-1})$$

then any formula for $\hat{g}_r$ immediately converts to a formula for g_r and vice versa.

Power sums

For $r \in \mathbb{Z}_{>0}$ define

$$p_r = p_r(x) = x_1^r + x_2^r + \cdots + x_n^r$$

Using

$$\frac{1}{1-ux_iz} = 1 + ux_iz + u^2x_i^2z^2 + \cdots$$

$$\frac{1-tx_iz}{1-ux_iz} = 1 + \frac{(u-t)x_iz}{1-ux_iz}$$

$$= 1 + (u-t)x_iz(1 + ux_iz + u^2x_i^2z^2 + \cdots)$$

then

$$\prod_{i=1}^{n} \frac{1-tx_iz}{1-ux_iz} = \left(\frac{1-tx_1z}{1-ux_1z}\right) \cdots \left(\frac{1-tx_nz}{1-ux_nz}\right)$$

gives

$$\hat{q}_r = \sum_{1 \le i_1 \le \cdots \le i_r \le n} (u-t)^{1+\#\{j \mid i_j < i_{j+1}\}} u^{\#\{j \mid i_j = i_{j+1}\}} x_{i_1}x_{i_2}\cdots x_{i_r}$$

Then

$$ev_{t=u}\left(\frac{1}{(u-t)}\hat{q}_r\right) = \sum_{i_1 = i_2 = \cdots = i_r} x_{i_1}\cdots x_{i_r} = p_r.$$

So

p_r is a specialization of $\hat{q}_r$

Monomial symmetric functions

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For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ the monomial symmetric function m_λ is

$$m_\lambda = \sum_{\gamma \in S_n\lambda} x^\gamma, \quad \text{where } x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$$

and S_n acts on $\mathbb{Z}^n$ by

$$w(\lambda_1, \ldots, \lambda_n) = (\lambda_{w(1)}, \ldots, \lambda_{w(n)}).$$

for $w \in S_n$ and $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$.

The q-binomial theorem gives

$$\frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} = \sum_{r \in \mathbb{Z}_{\geq 0}} \frac{(t;q)_r}{(q;q)_r} x_i^r z^r.$$

Using the q-binomial theorem on the product

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} = \frac{(tx_1z;q)_\infty}{(x_1z;q)_\infty} \cdots \frac{(tx_nz;q)_\infty}{(x_nz;q)_\infty}$$

gives

$$\mathring{g}_r = \sum_{|\mu| = r} u^r \frac{(tu^{-1};q)_\mu}{(q;q)_\mu} m_\mu$$

where

$$\frac{(tu^{-1};q)_\mu}{(q;q)_\mu} = \frac{(tu^{-1};q)_{\mu_1}}{(q;q)_{\mu_1}} \frac{(tu^{-1};q)_{\mu_2}}{(q;q)_{\mu_2}} \cdots \frac{(tu^{-1};q)_{\mu_\ell}}{(q;q)_{\mu_\ell}}.$$

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if $\mu = (\mu_1, \ldots, \mu_\ell)$. So

$$\hat{q}_r = \sum_{|\mu|=r} u^{r-\ell(\mu)} (u-t)^{\ell(\mu)} m_\mu,$$

$$g_r = \sum_{|\mu|=r} \frac{(t;q)_\mu}{(q;q)_\mu} m_\mu,$$

$$q_r = \sum_{|\mu|=r} (1-t)^{\ell(\mu)} m_\mu,$$

$$h_r = \sum_{|\mu|=r} m_\mu,$$

$$e_r = m_{(1^r)},$$

$$p_r = m_{(r)}.$$