06.08.2025
Adv. Disc. Math
Lect 5 A. Ram

Let $n \in \mathbb{Z}_{>0}$.

The <u>transpositions</u>, or <u>reflections</u> are

$$s_{ij} = 1 + E_{ij} + E_{ji} - E_{ii} - E_{jj}$$

for $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

The <u>simple transpositions</u> are

$$s_1 = s_{12}, \quad s_2 = s_{23}, \quad \ldots, \quad s_{n-1} = s_{n-1,n}.$$

The <u>symmetric group S_n</u> is the group generated by $s_1, s_2, \ldots, s_{n-1}$ under matrix multiplication.

the <u>general linear group</u> is

$$GL_n(\mathbb{C}) = \left\{ A \in M_{n\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} \text{there exists } A^{-1} \in M_n(\mathbb{C}) \\ \text{such that} \\ \quad AA^{-1} = 1 \text{ and } A^{-1}A = 1 \end{array} \right\}$$

with matrix multiplication.

<u>Proposition</u> The maps

$$\begin{array}{ccc} GL_m(\mathbb{C}) \times GL_n(\mathbb{C}) & \longrightarrow & GL_{m+n}(\mathbb{C}) \\ (A, B) & \longmapsto & \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) = A \oplus B \end{array}$$

and

$$S_n \times S_m \longrightarrow S_{n+m}$$
$$(v, w) \longmapsto \left(\begin{array}{c|c} v & 0 \\ \hline 0 & w \end{array}\right) = v \times w$$

are injective group homomorphisms.

Let $\gamma_1 = E_{11}$ in S_1 and

$$\gamma_k = E_{12} + E_{23} + \cdots + E_{k-1,k} + E_{k1} \quad \text{in } S_k$$

for $k \in \mathbb{Z}_{>1}$. For $\mu_1, \ldots, \mu_\ell \in \mathbb{Z}_{>0}$ let

$$\gamma_\mu = \gamma_{\mu_1} \times \cdots \times \gamma_{\mu_\ell} \quad \text{in } S_{\mu_1} \times \cdots \times S_{\mu_\ell} \subseteq S_{\mu_1 + \cdots + \mu_\ell}.$$

where $\mu = (\mu_1, \ldots, \mu_\ell)$.

A <u>Coxeter element of S_n</u> is an element of the conjugacy class of $\gamma_{(n)}$ in S_n

Let $[\gamma_\mu]$ denote the conjugacy class of γ_μ in $S_{\mu_1 + \cdots + \mu_\ell}$.

$$[\gamma_\mu] = \{ v^{-1} \gamma_\mu v \mid v \in S_{\mu_1 + \cdots + \mu_\ell} \}.$$

A <u>partition of n</u> is $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that $\ell \in \mathbb{Z}_{>0}$ and $\lambda_1, \ldots, \lambda_\ell \in \mathbb{Z}_{>0}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. and $\lambda_1 + \cdots + \lambda_\ell = n$.

<u>Theorem</u> (a) The map

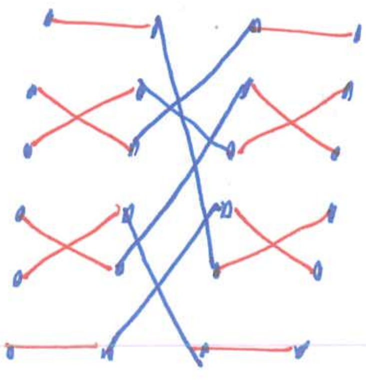
$$\begin{matrix} \left\{ \begin{matrix} \text{partitions} \\ \text{of } n \end{matrix} \right\} & \longleftrightarrow & \left\{ \begin{matrix} \text{conjugacy classes} \\ \text{of } S_n \end{matrix} \right\} \\ \lambda & \longmapsto & [\gamma_\lambda]. \end{matrix}$$

is a bijection.

(b) $\mathrm{Card}([\gamma_\lambda]) = \dfrac{n!}{z_\lambda}$, where

$z_\lambda = (1^{m_1} 2^{m_2} \cdots)(m_1! m_2! \cdots)$ if $\lambda = (1^{m_1} 2^{m_2} \cdots)$ so that m_i is the number of parts of size i in λ.

<u>Proof</u>



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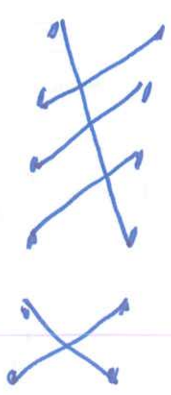

$= \gamma_{42}$

If

$$\gamma_\lambda = \gamma_1 \times \gamma_1 \times \gamma_1 \times \gamma_1 \times \gamma_2 \times \gamma_2 \times \gamma_2 \times \gamma_3 \times \gamma_4 \times \gamma_4$$

then

$$\mathrm{Card}(\mathrm{Stab}(\gamma_\lambda)) = 4! \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 3! \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2! \cdot 4 \cdot 4$$

$$= 4! \cdot 1^4 \cdot 3! \cdot 2^3 \cdot 1! \cdot 3^1 \cdot 2! \cdot 4^2$$

so that

$$\mathrm{Card}(\{\gamma_\lambda\}) = \frac{\mathrm{Card}(S_n)}{\mathrm{Card}(\mathrm{Stab}(\gamma_\lambda))} = \frac{n!}{z_\lambda}. \quad //.$$