Posets and Lattices

Two examples:

(1) The subset lattice.

Let $n \in \mathbb{Z}_{>0}$. The subset lattice is

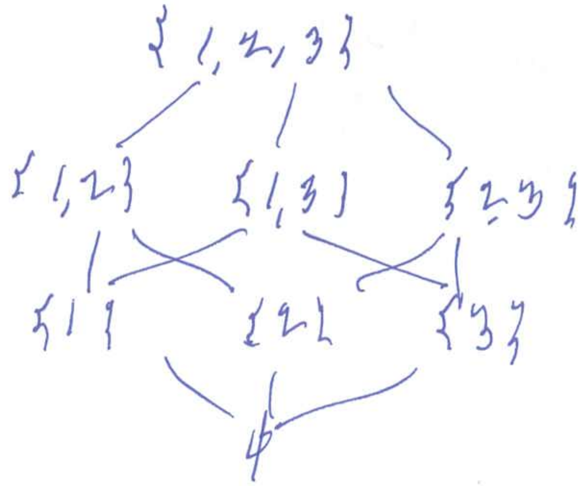
$$\mathcal{S}(n) = \{\text{subsets of } \{1, \ldots, n\}\}$$

with partial order given by inclusion.

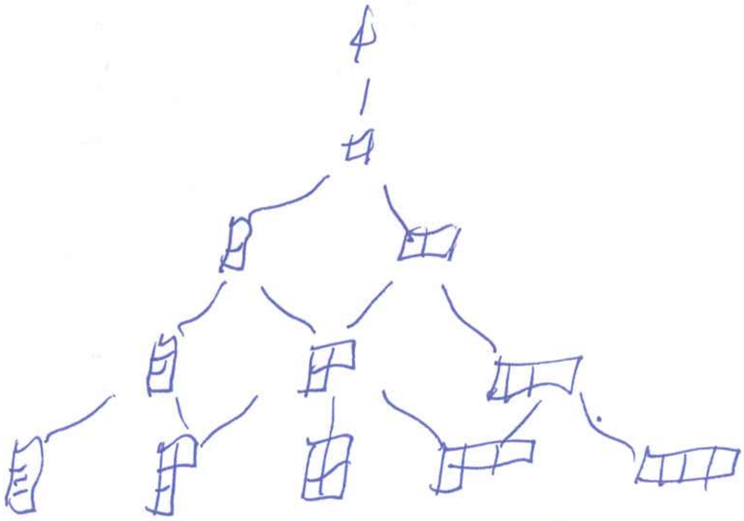
(2) The Young lattice is

$$\mathbb{Y} = \{\text{partitions}\}$$

with partial order given by inclusion.



The subset lattice $\mathcal{S}(3)$



Young lattice $\mathbb{Y}$

Then
$$\mathcal{S}(n) = \bigsqcup_{k=0}^{n} \mathcal{S}(n)_k \quad \text{where}$$

$$\mathcal{S}(n)_k = \left\{ \begin{array}{c} \text{subsets of } \{1, \ldots, n\} \\ \text{with } k \text{ element} \end{array} \right\}, \quad \mathrm{Card}(\mathcal{S}(n)_k) = \binom{n}{k}.$$

Posets and Lattices

Let S be a set.

A *relation on S* is a subset of $S \times S$.

Write $x \le y$ if $(x,y) \in \le$

A *partially ordered set* is a set P with a relation $\le$ on P such that

(a) If $x \in P$ then $x \le x$

(b) If $x, y, z \in P$ and $x \le y$ and $y \le z$ then $x \le z$.

(c) If $x, y \in P$ and $x \le y$ and $y \le x$ then $x = y$.

The *Hasse diagram of P* is the graph with

Vertices: P

Edges: $x \to y$ if $x < y$ and there does not exist $z \in P$ with $x < z < y$.

A *maximal chain in P* is a sequence $(x_1, x_2, \ldots, x_n)$ in P such that

(a) If $i \in \{1, \ldots, n-1\}$ then $x_i < x_{i+1}$ and there does not exist $z \in P$ with $x_i < z < x_{i+1}$

(b) There does not exist $z \in P$ such that $x_n < z$.

(c) There does not exist $z \in P$ such that $z < x_1$.

Lattices

Let P be a poset and $E \subseteq P$.

The infimum, or greatest lower bound, of E in P is $\ell \in P$ such that

(a) If $e \in E$ then $\ell \le e$,

(b) If $m \in P$ and m satisfies
if $e \in E$ then $m \le e$
then $m \le \ell$.

The supremum, or least upper bound, of E in P is $r \in P$ such that

(a) If $e \in E$ then $e \le r$

(b) If $t \in P$ and t satisfies
if $e \in E$ ~~and~~ then $e \le t$
then $r \le t$.

Notation If $k \in \mathbb{Z}_{>0}$ and $x_1, \ldots, x_k \in P$ use the notation
$$\inf(x_1, \ldots x_k) = \inf(\{x_1, \ldots x_k\}) \text{ and}$$
$$\sup(x_1, \ldots, x_k) = \sup(\{x_1, \ldots, x_k\}).$$

A lattice is a Poset P such that

if $x, y \in P$ then $\inf(x,y)$ and $\sup(x,y)$ exist in P.

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Modular Lattices

Let P be a lattice. Use the notation

$$x \wedge y = \inf(x,y) \quad \text{and} \quad x \vee y = \sup(x,y)$$

The language is $x \wedge y$ is "x meet y"
and $x \vee y$ is "x join y".

A <u>modular lattice</u> is a lattice P such that

if $m, n, p \in P$ and $p \le m$ then

$$m \vee (n \wedge p) = (m \vee p) \wedge n.$$

<u>Theorem</u> Let A be a $\mathbb{Z}$-algebra and let V be an A-module.

Let

$$\mathcal{G}(V) = \{A\text{-submodules of } V\}$$

partially ordered by inclusion.

Then

$$\mathcal{G}(V) \text{ is a modular lattice.}$$

<u>Proposition</u> Let A be a $\mathbb{Z}$-algebra and let V be an A-module.

Let $M, N, P \in \mathcal{G}(V)$.

(a) (infimums exist)

$$\inf(M,N) = M \cap N = \{v \in V \mid v \in M \text{ and } v \in N\}$$

(b) (supremums exist)

$$\sup(M,N) = M+N = \{m+n \mid m\in M \text{ and } n\in N\}$$

(c) (modular law) If $P\subseteq M$ then

$$M+(N\cap P) = (M+N)\cap P.$$

(d) (modular property)

$$\frac{M+N}{M} \cong \frac{N}{M\cap N}$$

<u>Proposition</u> Let $A=\mathbb{F}_q$ and let V be an A-module so that $V=\mathbb{F}_q^n$, where $n=\dim(V)$ as an $\mathbb{F}_q$-vector space.
Then

$$\mathbb{G}(\mathbb{F}_q^n) = \bigsqcup_{k=0}^{n} \mathbb{G}(\mathbb{F}_q^n)_k$$

where

$$\mathbb{G}(\mathbb{F}_q^n)_k = \left\{ \begin{matrix} \mathbb{F}_q\text{-subspaces } W \text{ of } \mathbb{F}_q^n \\ \text{with } \dim(W)=k \end{matrix} \right\}$$

and

$$\mathrm{Card}(\mathbb{G}(\mathbb{F}_q^n)_k) = \begin{bmatrix} n \\ k \end{bmatrix}$$