Advanced Discrete Math (1)
01.08.2025 Lect.3
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The symmetric group S_n

Let $n \in \mathbb{Z}_{>0}$.

The vector space of $n \times n$ matrices

$M_n(\mathbb{C})$ has $\mathbb{C}$-basis $\{E_{ij} \mid i,j \in \{1,\dots,n\}\}$

where E_{ij} is the matrix with 1 in the (i,j) entry and 0 elsewhere.

A permutation of n is $w \in M_{n\times n}(\mathbb{C})$ such that

(a) there is exactly one nonzero entry in each row and each column.

(b) The nonzero entries are 1.

The symmetric group is

$$S_n = \left\{ w \in M_{n\times n}(\mathbb{C}) \;\middle|\; \begin{matrix} w \text{ is a permutation} \\ \text{of } \{1,2,\dots,n\} \end{matrix} \right\}$$

with matrix multiplication.

Identify a permutation $w \in M_{n\times n}(\mathbb{C})$ with a bijection $w: \{1,\dots,n\} \to \{1,\dots,n\}$ by

$$w(i) = j \quad \text{if} \quad w_{ji} = 1,$$

where w_{ij} is the (i,j) entry of the matrix w.

The transpositions, or reflections in S_n are

$$s_{ij} = 1 + E_{ij} + E_{ji} - E_{ii} - E_{jj}, \quad \text{for } i,j \in \{1,\dots,n\} \text{ with } i \neq j$$

The simple transpositions are

$$s_1 = s_{12},\quad s_2 = s_{23},\quad \dots,\quad s_{n-1} = s_{n-1,n}$$

The general linear group is

$$GL_n(\mathbb{C}) = \left\{ A \in M_n(\mathbb{C}) \;\middle|\; \begin{matrix}\text{there exists } A^{-1} \in M_n(\mathbb{C}) \\ \text{with } AA^{-1} = 1 \text{ and } A^{-1}A = 1\end{matrix} \right\}$$

with matrix multiplication.

Proposition The maps

$$\begin{array}{ccc} GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) & \longrightarrow & GL_{n+m}(\mathbb{C}) \\ (A, B) & \longmapsto & \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) = A \oplus B \end{array}$$

and

$$\begin{array}{ccc} S_n \times S_m & \longrightarrow & S_{n+m} \\ (v, w) & \longmapsto & \left(\begin{array}{c|c} v & 0 \\ \hline 0 & w \end{array}\right) = v \times w \end{array}$$

are injective group homomorphisms.

Let $\gamma_1 = E_{11}$ in S_1 and

$$\gamma_k = E_{12} + E_{23} + \cdots + E_{k-1,k} + E_{k1} \text{ in } S_k$$

for $k \in \mathbb{Z}_{>1}$. For $\mu_1, \ldots, \mu_\ell \in \mathbb{Z}_{>0}$ let

$$\gamma_\mu = \gamma_{\mu_1} \times \cdots \times \gamma_{\mu_\ell} \text{ in } S_{\mu_1} \times \cdots \times S_{\mu_\ell} \subseteq S_{\mu_1 + \cdots + \mu_\ell}$$

A <u>Coxeter element</u> of S_n is an element of the conjugacy class of γ_n in S_n.

Let $[\gamma_\mu]$ denote the conjugacy class of γ_μ in $S_{\mu_1 + \cdots + \mu_\ell}$.

A <u>partition of n</u> is $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that $\lambda_1, \ldots, \lambda_\ell \in \mathbb{Z}_{>0}$ and $\lambda_1 \geq \cdots \geq \lambda_\ell$ and $\lambda_1 + \cdots + \lambda_\ell = n$.

<u>Theorem</u> (a) The map

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{partitions} \\ \text{of } n\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{conjugacy classes} \\ \text{of } S_n\end{array}\right\} \\ \lambda & \longmapsto & [\gamma_\lambda] \end{array}$$

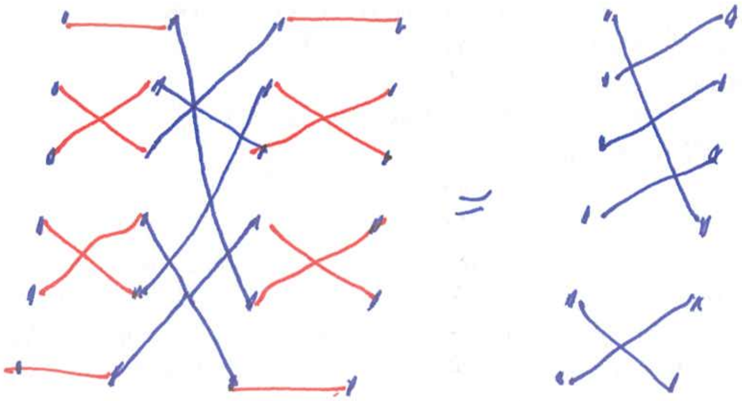
is a bijection.

(b) $\mathrm{Card}([\gamma_\lambda]) = \dfrac{n!}{z_\lambda}$, where

$$z_\lambda = (1^{m_1} 2^{m_2} \cdots)(m_1! \, m_2! \cdots) \quad \text{if } \lambda = (1^{m_1} 2^{m_2} \cdots)$$

so that m_i is the number of parts of size i in λ.

<u>Proof.</u>



$= \gamma_{42}$

and if

$$\gamma_\lambda = \gamma_1 \times \gamma_1 \times \gamma_1 \times \gamma_1 \times \gamma_2 \times \gamma_2 \times \gamma_2 \times \gamma_3 \times \gamma_4 \times \gamma_4$$

then

$$\mathrm{Card}(\mathrm{Stab}(\gamma_\lambda)) = 4! \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 3! \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2! \cdot 4 \cdot 4$$
$$= 4! \, 1^4 \cdot 3! \, 2^3 \cdot 1! \cdot 3^1 \cdot 2! \cdot 4^2$$

so that

$$\mathrm{Card}([\gamma_\lambda]) = \frac{\mathrm{Card}(S_n)}{\mathrm{Card}(\mathrm{Stab}(\gamma_\lambda))} = \frac{n!}{z_\lambda}.$$