Lecture 2 A. Ram

The binomial theorem and the exponential

For $k\in\mathbb{Z}_{\geq 0}$ define

$0!=1$ and $k!=k(k-1)\cdots 3\cdot 2\cdot 1$ for $k\in\mathbb{Z}_{>0}$.

For $k\in\{0,1,\ldots,n\}$ define

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

<u>Theorem</u> Let $n\in\mathbb{Z}_{>0}$ and $k\in\{1,\ldots,n\}$.

(a) Let S be a set with $\mathrm{Card}(S)=n$. Then

$$\binom{n}{k}=\left(\begin{matrix}\text{number of subsets of } S\\ \text{with cardinality } k\end{matrix}\right)$$

(b) $\binom{n}{k}$ is the coefficient of x^ky^{n-k} in $(x+y)^n$.

(c) If $k\in\{1,\ldots,n-1\}$ then

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$$

(d) In $\mathbb{C}[x,y]$,

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

Pascal's triangle

$$\begin{matrix} & & & & \binom{0}{0} & & & & \\ & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{matrix} = \begin{matrix} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{matrix}$$

Corollary (a) $f_{(k,(n-k))} = \binom{n}{k}$

(b) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(c) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ (see arXiv:0806.3940 page 3)

The rings of

formal power series $\mathbb{C}[[x]] = \{a_0 + a_1x + a_2x^2 + \cdots \mid a_i \in \mathbb{C}\}$

expressions $\mathbb{C}((x)) = \{a_{-\ell}x^{-\ell} + a_{-\ell+1}x^{-\ell+1} + \cdots \mid a_i \in \mathbb{C}\}$

polynomials

$\mathbb{C}[x] = \left\{ a_0 + a_1x + a_2x^2 + \cdots \;\middle|\; \begin{matrix} a_i \in \mathbb{C} \text{ and all but} \\ \text{a finite number of} \\ a_i \text{ are } 0 \end{matrix} \right\}$

31.07.2025 Adv. Disc. Math Lect. 2 A. Ram (3)

The exponential function

The exponential is

$$\exp(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Theorem

(a) If $xy = yx$ then
$$\exp(x+y) = \exp(x)\exp(y)$$

(b) $\frac{d}{dx}(\exp(x)) = \exp(x)$.

Theorem

(a) If $p \in \mathbb{C}[[x]]$ and $p(x+y) = p(x)p(y)$ then there exists $a \in \mathbb{C}$ such that $p(x) = \exp(ax)$.

(b) If $p \in \mathbb{C}[[x]]$ and $\frac{dp}{dx} = p$ then there exists $c_0 \in \mathbb{C}$ such that
$$p(x) = \cancel{\exp(c_0} \; c_0 \exp(x).$$

The binomial theorem

Define
$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{(a+k-1)!}{(a-1)!}$$

so that
$$n! = (1)_n \quad \text{and} \quad \binom{n}{k} = \frac{(k)_{n-k}}{(1)_k}$$

Define
$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

Define
$${}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}; q, z\right] = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(a_0;q)_k (a_1;q)_k \cdots (a_r;q)_k}{(q;q)_k (b_1;q)_k \cdots (b_r;q)_k} z^k$$

and
$${}_{r+1}F_r\left[\begin{matrix} \alpha_0, \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_r \end{matrix}; z\right] = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(\alpha_0)_k (\alpha_1)_k \cdots (\alpha_r)_k}{(1)_k (\beta_1)_k \cdots (\beta_r)_k} z^k$$

<u>Theorem</u> Let $\alpha \in \mathbb{C}$. Then
$$(1-z)^{-\alpha} = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(\alpha)_k}{k!} z^k = \sum_{k\in\mathbb{Z}_{\geq 0}} \binom{-\alpha}{k}(-z)^k = {}_1F_0[\alpha; z].$$

<u>Proof</u> (One option) Taylor series:
$$\frac{1}{k!}\frac{d^k}{dx^k}(1+x)^\alpha\Big|_{x=0} = \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k!}$$

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