Advanced Discrete Math 28.07.2015 (1)

Lecture 1.

A. Ram

Bratelli diagrams

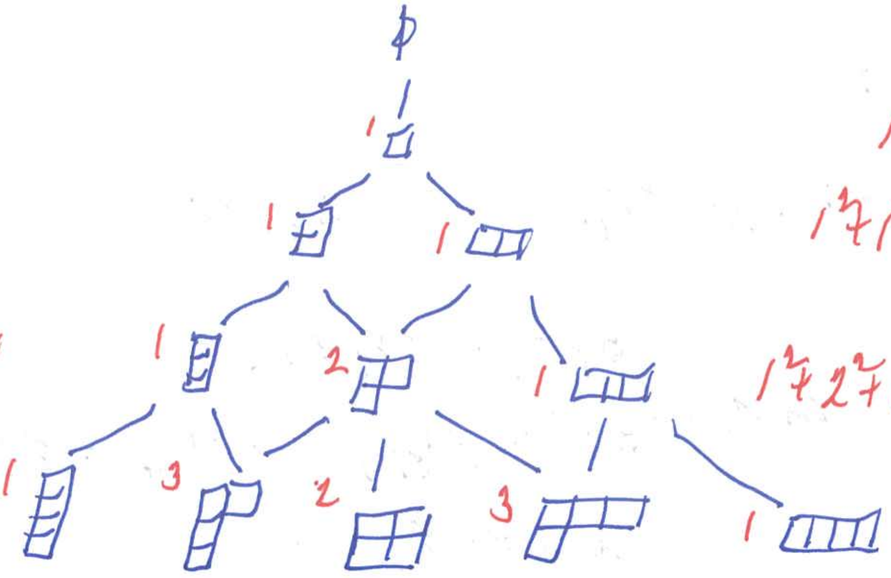
Young's lattice (boxes in a corner).

$1=1$ $\qquad$ $1^2=1$

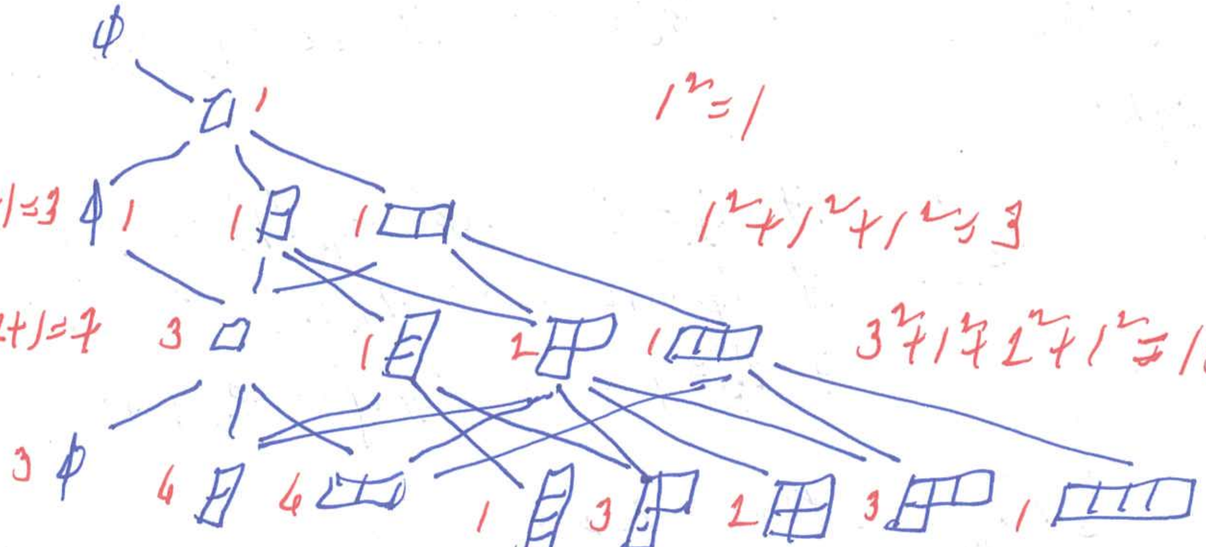
$1+1=2$ $\qquad$ $1^2+1^2=2$

$1+2+1=4$ $\qquad$ $1^2+2^2+1^2=6$

$1+3+2+3+1=10$ $\qquad$ $1^2+3^2+2^2+3^2+1=24$



Brauer Bratelli diagram (add and remove)



1 $\qquad$ $1^2=1$

$1+1+1=3$ $\qquad$ $1^2+1^2+1^2=3$

$3+1+2+1=7$ $\qquad$ $3^2+1^2+2^2+1^2=15$

$3+6+6+1+3+2+3+1=23$

$3^2+6^2+6^2+1^2+3^2+2^2+3^2+1^2=105$

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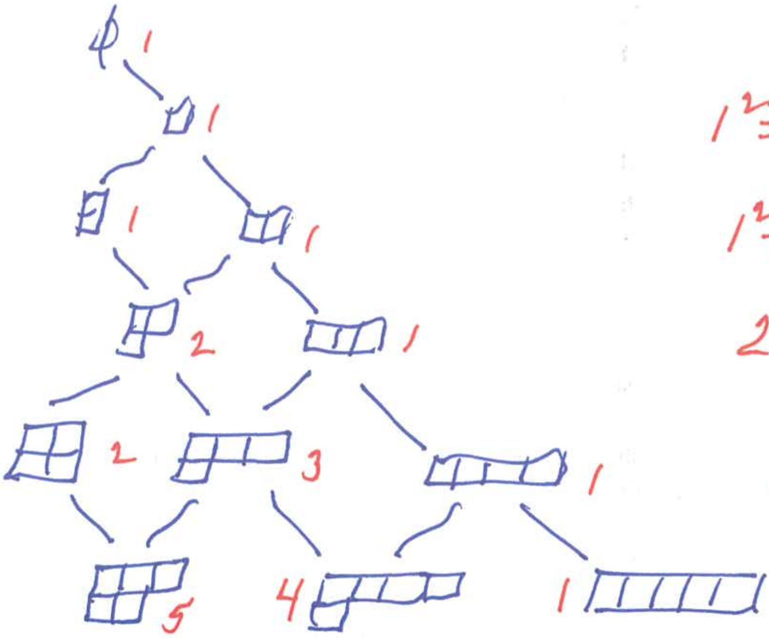
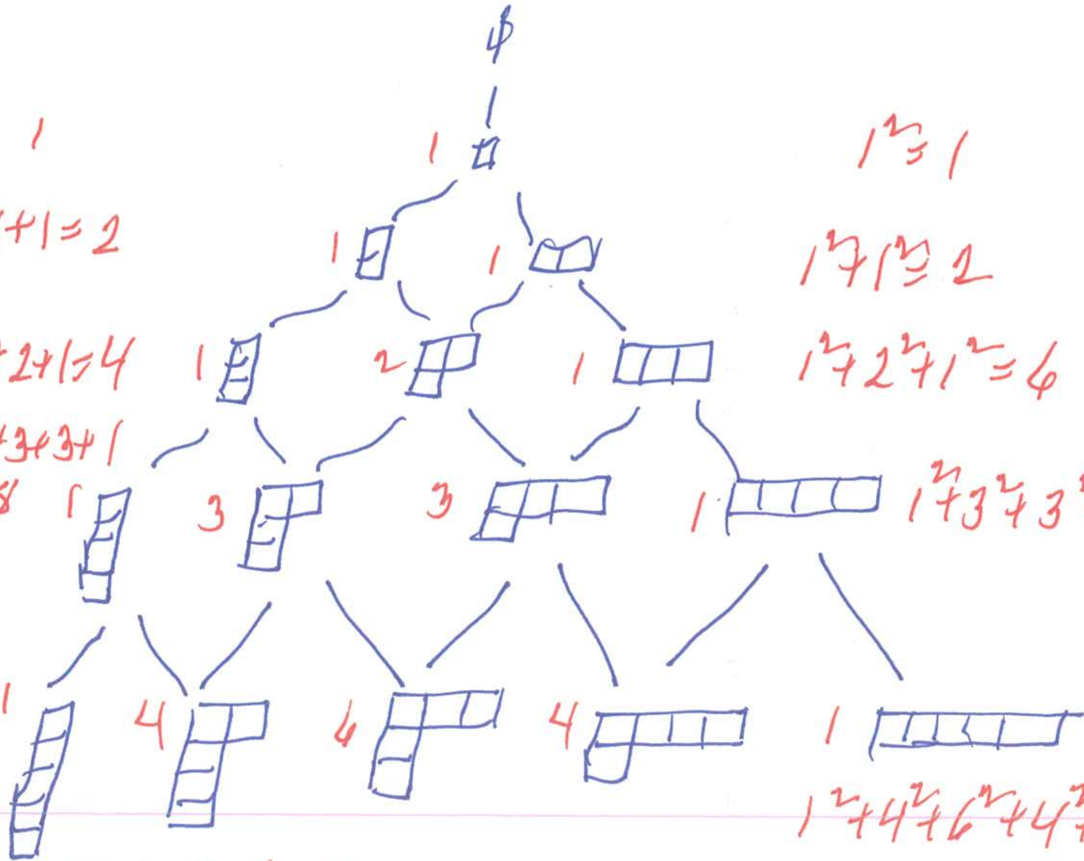
AdvDiscMath
Lect. 1 A. Ram

Temperley-Lieb Bratelli diagram
(restrict to two rows)



1	$1^2=1$
$1+1=2$	$1^2+1^2=2$
$2+1=3$	$2^2+1^2=5$
$2+3+1=6$	$2^2+3^2+1^2=14$
$5+4+1=10$	$5^2+4^2+1^2=42$

Pascal's triangle (restrict to along the wall)



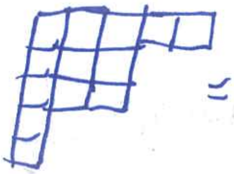
1	$1^2=1$
$1+1=2$	$1^2+1^2=2$
$1+2+1=4$	$1^2+2^2+1^2=6$
$1+3+3+1=8$	$1^2+3^2+3^2+1^2=20$
$1+4+6+4+1=16$	$1^2+4^2+6^2+4^2+1^2=70$

Partitions

A <u>partition</u> is $(\lambda_1,\dots,\lambda_\ell)$ with
$\ell\in\mathbb{Z}_{\geq 0}$, $\lambda_1,\dots,\lambda_\ell\in\mathbb{Z}_{>0}$ and $\lambda_1\geq\cdots\geq\lambda_\ell$.

A <u>box</u> is an element of $\mathbb{Z}^2$.

Identify $\lambda=(\lambda_1,\dots,\lambda_\ell)$ with a set of boxes so that λ has boxes $(r,1),(r,2),\dots,(r,\lambda_r)$ in row r.

$\lambda=(53311)=$  $=\left\{\begin{array}{l}(1,1),(1,2),(1,3),(1,4),(1,5),\\ (2,1),(2,2),(2,3),\\ (3,1),(3,2),(3,3),\\ (4,1)\\ (5,1)\end{array}\right\}$

Let
$$\ell(\lambda)=\ell \quad\text{and}\quad |\lambda|=\lambda_1+\cdots+\lambda_\ell$$
if $\lambda=(\lambda_1,\dots,\lambda_\ell)$. Write
$$\lambda\subseteq\mu \quad\text{if } \lambda \text{ is a subset of } \mu.$$

The <u>conjugate</u> of λ is
$$\lambda'=\{(c,r)\mid (r,c)\in\lambda\}.$$

For $n \in \mathbb{Z}_{\geq 0}$ let

$$\mathcal{Y}_n = \{\text{partitions } \lambda \text{ with } |\lambda| = n\} \quad \text{and}$$

$$\mathcal{Y} = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \mathcal{Y}_n$$

Let $\lambda \in \mathcal{Y}_n$. A <u>standard tableau of shape λ</u> is a function $T : \lambda \to \{1, \ldots, n\}$ such that

(a) If $(r,c), (r,c+1) \in \lambda$ then $T(r,c) < T(r,c+1)$

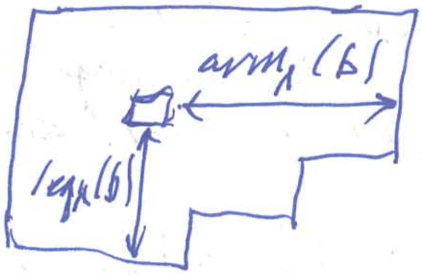
(b) If $(r,c), (r+1,c) \in \lambda$ then $T(r,c) < T(r+1,c)$.

Let

$$f_\lambda = \mathrm{Card}\left\{\begin{matrix}\text{standard tableaux } T \\ \text{of shape } \lambda\end{matrix}\right\}$$

If $(r,c) \in \lambda$ define

$$\mathrm{arm}_\lambda(r,c) = \lambda_r - c$$
$$\mathrm{leg}_\lambda(r,c) = \lambda'_c - r$$



<u>Theorem</u> Let $\lambda \in \mathcal{Y}_n$.

(a) $$f_\lambda = \frac{n!}{\prod_{b \in \lambda} (\mathrm{arm}_\lambda(b) + \mathrm{leg}_\lambda(b) + 1)}$$

(b) $$n! = \sum_{\lambda \in \mathcal{Y}_n} f_\lambda^2$$