

22.08.2025

The generators and relations for S_n

Adv. Discrete Math ①
Lect 12 A. Ram

Generators A: Permutation matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{array}{c} \text{Diagram of a permutation matrix} \end{array}$$

Relations A: $VW = U$ is given by matrix multiplication

$$\begin{array}{c} \text{Diagram of } V \text{ and } W \\ V \quad W \end{array} = \begin{array}{c} \text{Diagram of } U \end{array}$$

Generators B: $s_1, \dots, s_{n-1}$

Relations B $s_j^2 = 1$, $s_j s_k = s_k s_j$ if $k \notin \{j-1, j+1\}$
and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Generators B in terms of Generators A

$$s_i = \begin{array}{c} \text{Diagram of } s_i \end{array} \text{ for } i \in \{1, \dots, n-1\}$$

Relations 3 from relations A

$$s_i^{-n} = \overbrace{s_i s_i \dots s_i}^n = 1.$$

$$s_j s_k s_j = s_k s_j s_j = s_k$$

$$s_i s_j s_i = s_j s_i s_i = s_j \quad \text{and}$$

$$s_i s_j s_i s_j = s_j s_i s_j s_i = s_j s_i s_i s_j = s_j s_i$$

Generators A from Generators B (the subject of reduced words)

Let $w \in S_n$

A reduced word for w is an expression

$$w = s_{i_1} \dots s_{i_\ell} \quad \text{with } i_1, \dots, i_\ell \in \{1, \dots, n\}$$

and ℓ minimal. The length of w is $\ell(w)$, the length of a reduced word for w .