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Length and reduced words

Let $w \in S_n$.

A reduced word for w is an expression

$$w = s_{i_1} \cdots s_{i_\ell} \quad \text{with } i_1, \ldots, i_\ell \in \{1, \ldots, n-1\}$$

and ℓ minimal.

The length of w is $\ell(w)$, the length of a reduced word for w.

$$\begin{tikzcd} & s_1 \arrow[r, leftrightarrow] & s_1s_2 \arrow[dr, leftrightarrow] & \\ 1 \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & & & s_1s_2s_1 \\ & s_2 \arrow[r, leftrightarrow] & s_2s_1 \arrow[ur, leftrightarrow] & \end{tikzcd} = \begin{tikzcd} & (213) \arrow[r, leftrightarrow] & (231) \arrow[dr, leftrightarrow] & \\ (123) \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & & & (321) \\ & (132) \arrow[r, leftrightarrow] & (312) \arrow[ur, leftrightarrow] & \end{tikzcd}$$

Constructing a reduced word

Let w be an $n \times n$ permutation matrix.

Let $j_1 > 1$ be maximal such that $w_{j_1 1} \neq 0$. Set

$$w^{(1)} = \begin{cases} w, & \text{if } j_1 \text{ does not exist,} \\ s_1 \cdots s_{j_1 - 1} w, & \text{if } j_1 \text{ exists.} \end{cases}$$

Let $j_2 > 2$ be maximal such that $w^{(1)}_{j_2 2} \neq 0$. Set

$$w^{(2)} = \begin{cases} w^{(1)}, & \text{if } j_2 \text{ does not exist,} \\ s_2 \cdots s_{j_2 - 1}, & \text{if } j_2 \text{ exists} \end{cases}$$

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Continue this process to produce $w^{(1)}, w^{(2)}, \dots, w^{(n)}$. Then

$$w^{(n)} = 1 \quad \text{and} \quad w = \cdots (s_{j_2-1}\cdots s_2)(s_{j_1-1}\cdots s_1)$$

is a reduced word for w.

<u>Constructing a reduced word for $g \in GL_n(\mathbb{F})$</u>

Let $g \in GL_n(\mathbb{F})$. Let $g(i,j)$ be the (i,j) entry of g.
Let

$$y_i(c) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \quad \text{for } i \in \{1, \dots, n-1\} \text{ and } c \in \mathbb{F}.$$

Let $j_1 \geq 1$ ~~$\in \{1, 2, \dots, n-1\}$~~ be maximal such that $g(j_1, 1) \neq 0$.
Let

$$g^{(1)} = \begin{cases} g, & \text{if } j_1 \text{ does not exist}, \\ y_1\left(\dfrac{g(1,1)}{g(j_1,1)}\right)^{-1} y_2\left(\dfrac{g(1,2)}{g(j_1,1)}\right)^{-1} \cdots y_{j_1-1}\left(\dfrac{g(j_1-1,1)}{g(j_1,1)}\right)^{-1} g, & \text{if } j_1 \text{ exists}. \end{cases}$$

Let $j_2 \geq 2$ be maximal such that $g(j_2, 2) \neq 0$.
Let

$$g^{(2)} = \begin{cases} g^{(1)}, & \text{if } j_2 \text{ does not exist}, \\ y_2\left(\dfrac{g^{(1)}(2,2)}{g^{(1)}(j_2,2)}\right)^{-1} \cdots y_{j_2-1}\left(\dfrac{g^{(1)}(j_2-1,2)}{g^{(1)}(j_2,2)}\right)^{-1} g^{(1)}, & \text{if } j_2 \text{ exists}. \end{cases}$$

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Continue this process to produce

$$g^{(1)}, g^{(2)}, \ldots, g^{(n)}.$$

Then $g^{(n)}$ has the property that
the first nonzero entry in row $j+1$
is to the right of the first nonzero entry in
row j.

If g is invertible then $g^{(n)}$ is upper triangular.

Let

$$b = g^{(n)}.$$

Let

$$u = h_n(b(n,n))^{-1} \cdots h_1(b(1,1))^{-1}$$

Then

$$u = x_{n-1,n}(u(n-1,n))\, x_{n-2,n}(u(n-2,n)) \cdots x_{1n}(u(1,n))$$
$$\cdot\, x_{n-2,n-1}(u(n-2,n-1)) \cdots x_{1,n-1}(u(1,n-1))$$
$$\cdot \cdots$$
$$\cdot\, x_{21}(u(2,1)).$$

Then

$$g = \cdots \left(y_{j_2-1}\left(\frac{g^{(1)}(j_2-1,2)}{g^{(1)}(j_2,2)}\right) \cdots y_2\left(\frac{g^{(1)}(2,2)}{g^{(1)}(j_2,2)}\right)\right)$$
$$\cdot \left(y_{j-1}\left(\frac{g(j-1,1)}{g(j,1)}\right) \cdots y_1\left(\frac{g(1,1)}{g(j,1)}\right)\right)$$
$$\cdot\, h_1(b(1,1)) \cdots h_n(b(n,n)) \cdot u$$

is a reduced word for g.