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Adv. Disc. Math
Lect. 11 D. Ram.

Length and reduced words

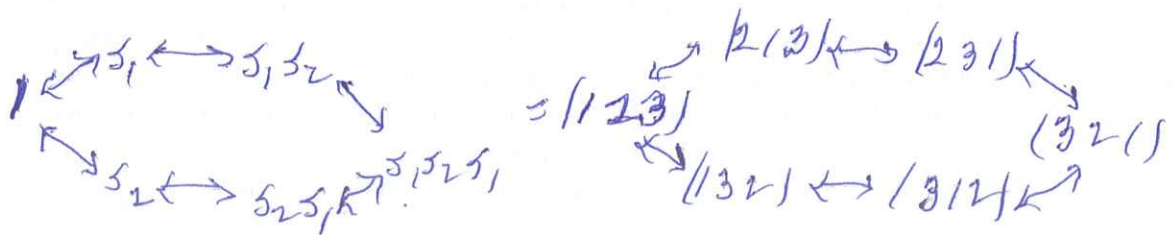
Let $w \in S_n$.

1) reduced work for w is an expression

$$W \subseteq \mathcal{Z}_1 \cup \mathcal{Z}_2 \quad \text{with} \quad \mathcal{Z}_1, \mathcal{Z}_2 \subseteq \{1, \dots, n-1\}$$

and a minimal.

The length of w is $\ell(w)$, the length of a reduced word for w .



Constructing a reduced word

Let w be an $n \times n$ permutation matrix.

Let $j, \geq 1$ be minimal such that $w_{j,1} \neq 0$. Set

$$w^{(i)} = \begin{cases} w, & \text{if } j_i \text{ does not exist,} \\ s_1 \cdots s_{j_i-1} w, & \text{if } j_i \text{ exists.} \end{cases}$$

Let $j_2 \geq 2$ be maximal such that $w_{j_2}^{(1)} \neq 0$. Set

$$w^{(H)} = \begin{cases} w^{(U)}, & \text{if } j_r \text{ does not exist,} \\ 3n \cdot 5_{j_r}, & \text{if } j_r \text{ exists} \end{cases}$$

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(2)

Continue this process to produce
 $w^{(1)}, w^{(2)}, \dots, w^{(n)}$, then

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$w^{(n)} = 1$ and $w = \dots (s_{j_{n-1}} \dots s_{j_2})(s_{j_1} \dots s_1)$
 is a reduced word for w .

Constructing a reduced word for $g \in GL_n(\mathbb{F})$

Let $g \in GL_n(\mathbb{F})$. Let $g(i, j)$ be the (i, j) entry of g .

Let

$$g_i(L) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & & c_i \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \text{ for } i \in \{1, \dots, n\} \text{ and } c_i \in \mathbb{F}.$$

Let $j_1 \in \{1, 2, \dots, n\}$ be maximal such that $g(j_1, 1) \neq 0$.

Let

$$g^{(1)} = \begin{cases} g, & \text{if } j_1 \text{ does not exist,} \\ g_1 \left(\frac{g(1,1)}{g(j_1,1)} \right)^{-1} g_2 \left(\frac{g(1,2)}{g(j_1,1)} \right)^{-1} \dots g_{j_1-1} \left(\frac{g(1,j_1-1)}{g(j_1,1)} \right)^{-1} g, & \text{if } j_1 \text{ exists.} \end{cases}$$

Let $j_2 > 2$ be maximal such that $g(j_2, 2) \neq 0$.

Let

$$g^{(n)} = \begin{cases} g^{(1)}, & \text{if } j_n \text{ does not exist,} \\ g_2 \left(\frac{g^{(1)}(2,2)}{g^{(1)}(j_n,2)} \right)^{-1} \dots g_{j_2-1} \left(\frac{g^{(1)}(j_2-1,2)}{g^{(1)}(j_n,2)} \right)^{-1} g^{(1)}, & \text{if } j_n \text{ exists.} \end{cases}$$

Continue this process to produce

$$g^{(1)}, g^{(2)}, \dots, g^{(n)}.$$

Then $g^{(n)}$ has the property that

the first nonzero entry on row $j+1$
is to the right of the first nonzero entry in
row j .

If g is invertible then $g^{(n)}$ is upper triangular.

Let

$$b = g^{(n)}.$$

Let

$$u = h_n(b(n,n))^{-1} \dots h_1(b(1,1))^{-1}$$

Then

$$u = x_{n-1,n}(u(n-1,n)) x_{n-2,n}(u(n-2,n)) \dots x_{1,n}(u(1,n))$$

$$\cdot x_{n-2,n-1}(u(n-2,n-1)) \dots x_{1,n-1}(u(1,n-1))$$

$$\cdot \dots$$

$$\cdot x_{2,1}(u(2,1)).$$

Then

$$g = \dots (y_{n-1} \left(\frac{g^{(1)}(j_{n-1,2})}{g^{(1)}(j_{n-1,1})} \dots y_2 \left(\frac{g^{(1)}(2,2)}{g^{(1)}(j_{n-1,2})} \right) \right.$$

$$\cdot (y_{j-1} \left(\frac{g^{(1)}(j,1)}{g^{(1)}(j,1)} \dots y_1 \left(\frac{g^{(1)}(1,1)}{g^{(1)}(j,1)} \right) \right.$$

$$\cdot h_1(b(1,1)) \dots h_n(b(n,n)) \cdot u$$

is a reduced word for g .