The Hecke algebra

Let

$\mathbb{C}[G/B]$ be the vector space with basis

$\mathcal{F}(\mathbb{F}_q^n) = \{$maximal chains in $\mathcal{G}(\mathbb{F}_q^n)\}$.

The group $G = GL_n(\mathbb{F}_q)$ acts on $\mathbb{C}[G/B]$ by the $\mathbb{C}$-linear maps given by

$$g(0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{F}_q^n) = (0 \subsetneq gV_1 \subsetneq gV_2 \subseteq \cdots \subseteq gV_n = \mathbb{F}_q^n)$$

for $g \in GL_n(\mathbb{F}_q)$

For $i \in \{1, \dots, n-1\}$ define a $\mathbb{C}$-linear map

$T_i : \mathbb{C}[G/B] \to \mathbb{C}[G/B]$ by

$$T_i(0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{\substack{W \neq V_i \\ V_{i-1} \subsetneq W \subsetneq V_{i+1}}} (0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_{i+1} \subsetneq \cdots \subseteq V_n).$$

Then

$$T_j^2 = (q-1)T_j + q, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$T_j T_k = T_k T_j \quad \text{and} \quad g T_j = T_j g$$

for $i \in \{1, \dots, n-2\}$, $k, j \in \{1, \dots, n-1\}$ with $k \in \{j-1, j+1\}$ and $g \in GL_n(\mathbb{F}_q)$.

<u>The symmetric group</u>

Let $\mathbb{C}S_n$ be the vector space with basis

$F(\mathbb{F}_1^n) = F(S(n))$.

For $i \in \{1, \dots, n-1\}$ define a linear transformation

$s_i : \mathbb{C}S_n \to \mathbb{C}S_n$ by

$$s_i(0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{\substack{W \neq V_i \\ V_{i-1} \subsetneq W \subsetneq V_{i+1}}} (0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_{i+1} \subsetneq \cdots \subsetneq V_n)$$

Then

$$s_j^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_j s_k = s_k s_j \quad \text{and} \quad g s_j = s_j g$$

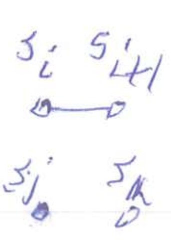
for $i \in \{1, \dots, n-2\}$, $j, k \in \{1, \dots, n-1\}$ with $k \notin \{j-1, j+1\}$ and $g \in S_n$.

There are <u>two</u> <u>different</u> actions of S_n here.

<u>Theorem</u> The symmetric group S_n is presented by generators $s_1, \dots, s_{n-1}$ and relations

$$s_j^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_j s_k = s_k s_k$$



for $k, j \in \{1, \dots, n-1\}$ ~~and~~ with $k \notin \{j-1, j+1\}$

and $i \in \{1, \dots, n-2\}$