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ADM ①
L6
WM3

The subset lattice $\mathfrak{S}(n)$

$$\mathfrak{S}(n) = \{ \text{subsets of } \{1,\dots,n\}\}$$

partially ordered by inclusion

Then $\mathfrak{S}(n)$ is a ranked modular lattice

$$\mathfrak{S}(n) = \bigsqcup_{k=0}^{n} \mathfrak{S}(n)_k \,, \quad \text{where}$$

$$\mathfrak{S}(n)_k = \left\{ \begin{array}{l} \text{subsets } V \subseteq \{1,\dots,n\} \\ \text{with } \mathrm{Card}(V) = k \end{array} \right\}$$

Then

$$\mathrm{Card}(\mathfrak{S}(n)_k) = \binom{n}{k} \quad \text{and} \quad \sum_{k=0}^{n} x^k \binom{n}{k} = (1+x)^n.$$

Proposition $\mathrm{Aut}(\mathfrak{S}(n)) = S_n$

Let $E_k = \{1,\dots,k\}$.

Proposition (a) $\mathrm{Stab}_{S_n}(E_k) = S_k \times S_{n-k}$ and

$$\mathrm{Stab}_{S_n}(\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_n) = \{1\} = S_1 \times \cdots \times S_1$$

(b) $\mathfrak{S}(n)_k \cong S_n / S_k \times S_{n-k}$ and $\mathcal{F}(\mathfrak{S}(n)) \cong S_n/\{1\}$

(c) $\mathfrak{S}(n)_k \cong \mathfrak{S}(n)_{n-k}$.

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Maximal chains in $\mathfrak{B}(n)$

Proposition The map

$$\mathcal{F}(\mathfrak{B}(n)) \longrightarrow S_n$$

$$(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) \longmapsto (V_1, V_2 - V_1, \ldots, V_n - V_{n-1})$$

is a bijection.

Let $\mathbb{C}S_n$ be the vector space with basis indexed by the elements of $\mathcal{F}(\mathfrak{B}(n))$.
For $i \in \{1, \ldots, n-1\}$ define a $\mathbb{C}$-linear transformation $s_i : \mathbb{C}S_n \to \mathbb{C}S_n$ by

$$s_i(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{V_{i-1} \subsetneq W \subsetneq V_{i+1}} (\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subseteq W \subseteq V_{i+1} \subsetneq \cdots \subsetneq V_n)$$

Then

$$s_i s_j = s_j s_i \quad \text{if } j \notin \{i-1, i+1\}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_i^2 = 1$$

$$g s_i = s_i g, \quad \text{for } g \in S_n,$$

where

$$g(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = (\emptyset \subsetneq gV_1 \subsetneq \cdots \subsetneq gV_n)$$

for $g \in S_n$ and $(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) \in \mathcal{F}(\mathfrak{B}(n))$.

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Simple reflections

Let $s_i = 1 + E_{i,i+1} + E_{i+1,i} - E_{ii} - E_{i+1,i+1}$,
for $i \in \{1, \dots, n-1\}$

Theorem The symmetric group S_n is presented by generators $s_1, \dots, s_{n-1}$ and relations

$s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$
if $j \notin \{i-1, i+1\}$.

The proof requires four steps:

(1) Generators A in terms of Gens B.
(2) Gens B in terms of Gens A
(3) Relations A from Relations B
(4) Relations B from Relations A.

Here

Gens A: {permutation matrices}

Relations A: matrix multiplication of permutation matrices.

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Reduced words

Let $w \in S_n$. A <u>reduced word for w</u> is an expression

$$w = s_{i_1} \cdots s_{i_\ell} \quad \text{with } i_1, \dots, i_\ell \in \{1, \dots, n-1\}$$

with ℓ minimal.

Let $j_1 \geq 1$ be minimal such that $w_{j_1,1} \neq 0$.

Let $w^{(1)} = s_1 \cdots s_{j_1-1} w$. and $w^{(1)} = w$ if j_1 does not exist

Let $j_2 \geq 2$ be minimal such that $w^{(1)}_{j_2,2} \neq 0$

Let $w^{(2)} = s_2 \cdots s_{j_2-2} w^{(1)}$ and $w^{(2)} = w^{(1)}$ if j_2 does not exist.

Then $w^{(n)} = 1$ and

$$w = \cdots (s_{j_2-2} \cdots s_2)(s_{j_1-1} \cdots s_1)$$

is a reduced word for w.

~~Let~~ The length of w is $\ell(w)$ the length of a reduced word for w.

<u>Proposition</u> Let

$$\mathrm{Inv}(w) = \{ (i,j) \mid i,j \in \{1, \dots, n\} \text{ with } i<j \text{ and } w(i) > w(j) \}$$

Then $\ell(w) = \mathrm{Card}(w)$.

Reduced words continued 06.11.2024 ADM. L6 W2L3 (5)

Define a graph $\Gamma(w)$ with

Vertices: {reduced words of w}

Edges: $u \to u'$ if $u' = s_{i_1'} \cdots s_{i_k'}$ is obtained from u by applying ~~the~~ a relation

$s_i s_j = s_j s_i$ or $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

Proposition Let $w \in S_n$. Then $\Gamma(w)$ is connected.