05.11.2024
ADM ①
L5
W2L2

The subspace lattice $G(\mathbb{F}^n)$

The subspace lattice of $\mathbb{F}^n$ is

$$G(\mathbb{F}^n) = \{\mathbb{F}\text{-subspaces of } \mathbb{F}^n\}$$

partially ordered by inclusion.

Then $G(\mathbb{F}^n)$ is a ranked modular lattice

$$G(\mathbb{F}^n) = \bigsqcup_{k=0}^{n} G(\mathbb{F}^n)_k, \quad \text{where}$$

$$G(\mathbb{F}^n)_k = \left\{ \begin{array}{l} \mathbb{F} \text{ subspaces } V \subseteq \mathbb{F}^n \\ \text{with } \dim(V) = k \end{array} \right\}.$$

Theorem Let $\mathbb{F}_q$ be a finite field with q elements. For $r \in \mathbb{Z}_{>0}$ let

$$[r] = \frac{q^r - 1}{q-1}, \quad [r]! = [r][r-1]\cdots[2][1]$$

and for $k \in \{0, 1, \ldots, n\}$ let

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

Then

$$\mathrm{Card}(G(\mathbb{F}_q^n)_k) = \begin{bmatrix} n \\ k \end{bmatrix} \quad \text{and}$$

$$\sum_{k=0}^{n} x^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} = (1+x)(1+xq)\cdots(1+xq^{n-1})$$

$$= (-x; q)_n$$

Automorphisms of $\mathbb{G}(\mathbb{F}^n)$

A <u>morphism of posets</u> is a function $f: P \to Q$ such that

if $x, y \in P$ and $x \leq y$ then $f(x) \leq f(y)$

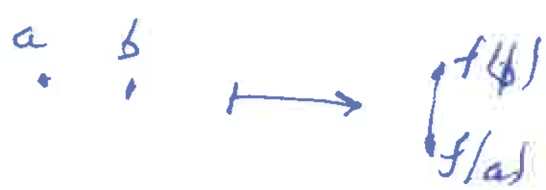
An <u>isomorphism</u> of posets is a morphism $f: P \to Q$ such that the inverse function $f^{-1}: Q \to P$ exists and f^{-1} is a morphism of posets.

An <u>automorphism</u> of P is an isomorphism $f: P \to P$ of posets.

<u>Proposition</u> $\mathrm{Aut}(\mathbb{G}(\mathbb{F}^n)) = GL_n(\mathbb{F})$,

where $GL_n(\mathbb{F}) = \{ g \in M_n(\mathbb{F}) \mid g^{-1} \text{ exists in } M_n(\mathbb{F}) \}$.

<u>HW</u> Give an example of a morphism $f: P \to Q$ of finite posets that is bijective and is not an isomorphism of posets.



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Projective space and cosets

Let $\mathbb{F}$ be a field and define an equivalence relation on $\mathbb{F}^n - \{(0,\dots,0)\}$

$$[a_1,\dots,a_n] = [\lambda a_1,\dots,\lambda a_n] \quad \text{if } a_1,\dots,a_n\in\mathbb{F},\ \lambda\in\mathbb{F}^\times$$

The projective space $\mathbb{P}^{n-1}$ is

$$\mathbb{P}^{n-1} = \{\text{equivalence classes}\}.$$

Let $\{e_1,\dots,e_n\}$ be an $\mathbb{F}$-basis of $\mathbb{F}^n$ and let $E_k = \mathbb{F}\text{-span}\{e_1,\dots,e_k\}$ for $k\in\{0,\dots,n\}$.

Proposition Let $G = GL_n(\mathbb{F})$ acting on $\mathcal{G}(\mathbb{F}^n)$. Then

(a) $\mathrm{Stab}_G(E_k) = P_k = \left\{ \begin{array}{cc} k & n-k \\ \end{array} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ and

$$\mathrm{Stab}_G(0\subsetneq E_1\subsetneq\cdots\subsetneq E_n) = B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$$

(b) $\mathcal{G}(\mathbb{F}^n)_k \cong G/P_k$ and $\mathcal{F}(\mathcal{G}(\mathbb{F}^n)) \cong G/B$

where $\mathcal{F}(\mathcal{G}(\mathbb{F}^n)) = \{\text{maximal chains in } \mathcal{G}(\mathbb{F})\}$

(c) ~~Let $\mathbb{F}_q$ be a finite~~

$$\mathcal{G}(\mathbb{F}^n)_1 \cong \mathbb{P}^{n-1} \quad \text{and} \quad \mathcal{G}(\mathbb{F}^n) \cong \mathbb{P}^{n-1}$$

and $\mathcal{G}(\mathbb{F}^n)_k \cong \mathcal{G}(\mathbb{F}^n)_{n-k}$.

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Counting and the Hecke algebra

Let $\mathbb{F}_q$ be a finite field with q elements.

Proposition

$$\mathrm{Card}(GL_n(\mathbb{F}_q)) =$$

$$\mathrm{Card}(B) = q^{\frac{n(n-1)}{2}} (q-1)^n$$

$$\mathrm{Card}(\mathcal{F}/\mathcal{B}(\mathbb{F}_q^n)) = [n]!$$

$$\mathrm{Card}(\mathbb{P}^1) = 1+q \quad \text{and} \quad \mathrm{Card}(\mathbb{P}^{n-1}) = 1+q+\cdots+q^{n-1}.$$

Let $\mathbb{C}[G/B]$ be a $\mathbb{C}$-vector space with basis indexed by the elements of $\mathcal{F}/\mathcal{B}(\mathbb{F}_q^n)$.

For $i \in \{1, \dots, n-1\}$ define a linear transformation $T_i : \mathbb{C}[G/B] \to \mathbb{C}[G/B]$ by

$$T_i(\phi \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{V_{i-1} \subsetneq W \subsetneq V_{i+1}} (\phi \subseteq V_1 \subseteq \cdots \subseteq V_{i-1} \subseteq W \subseteq V_{i+1} \subseteq \cdots \subseteq V_n)$$

Then

$$T_i T_j = T_j T_i \quad \text{if} \quad j \notin \{i-1, i+1\}$$

$$T_i T_{i+1} T_i = T_i T_{i+1} T_i$$

$$T_i^2 = (q-1) T_i + q$$

$$g T_i = T_i g \quad \text{for} \quad g \in GL_n(\mathbb{F}_q).$$

$$(g(\phi \subsetneq V_1 \subsetneq \cdots \subsetneq V_n)) = (\phi \subsetneq gV_1 \subseteq \cdots \subseteq gV_n)).$$