31.10.2024 (1)
ADM. L4
W2L1

Posets and lattices

Two examples:

(1) The subset lattice

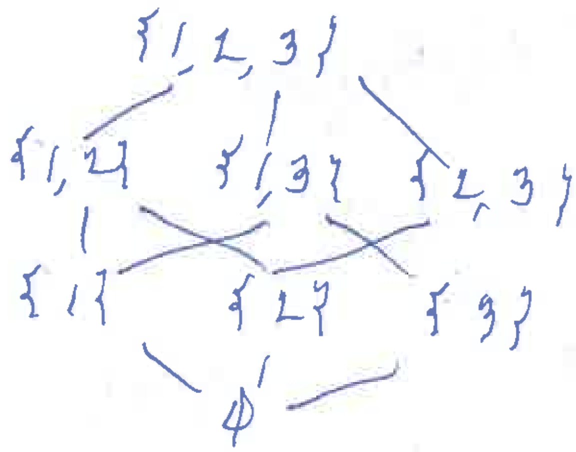
Let $n \in \mathbb{Z}_{>0}$. The <u>subset lattice</u> is

$$S_n = \{\text{subsets of } \{1, \dots, n\}\}$$

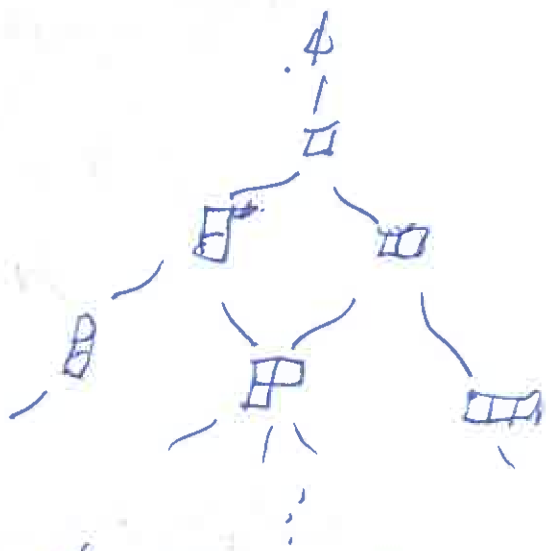
with partial order given by inclusion.

(2) The <u>Young lattice</u> is

$\mathcal{Y} = \{\text{partitions}\}$ with partial order given by inclusion.



The subset lattice S_3



The Young lattice $\mathcal{Y}$

Then

$$S(n) = \bigsqcup_{k=0}^{n} S(n)_k \qquad S(n)_k = \left\{\begin{array}{l}\text{subsets of } \{1, \dots, n\} \\ \text{with cardinality } k\end{array}\right\}$$

with $\mathrm{Card}(S(n)_k) = \binom{n}{k}$

and

$$\mathcal{Y} = \bigsqcup_{k=0}^{\infty} \mathcal{Y}_k \quad \text{where } \mathcal{Y}_k = \left\{\begin{array}{l}\text{partitions} \\ \text{with } k \text{ boxes}\end{array}\right\}$$

31.10.2024
ADM

(1)
L4
W2L7

Posets and lattices

Let S be a set.

A <u>relation</u> on S is a subset of $S \times S$

Write $x \leq y$ if $(x,y) \in \leq$.

A <u>partially ordered set</u> is a set P with a relation $\leq$ on P such that

(a) If $x \in P$ then $x \leq x$

(b) If $x, y, z \in P$ and $x \leq y$ and $y \leq z$ then $x \leq z$

(c) If $x, y \in P$ and $x \leq y$ and $y \leq x$ then $x = y$.

The <u>Hasse diagram</u> of P is the graph with

Vertices: P

Edges: $x \to y$ if $x \leq y$

A <u>maximal chain</u> in P is a function

$$\begin{matrix} \mathbb{Z}_{>0} \to P \\ i \mapsto x_i \end{matrix} \quad \text{such that}$$

(a) If $i \in \mathbb{Z}_{>0}$ then $x_i < x_{i+1}$

(b) There does not exist $y \in P$ such that $x_i < y < x_{i+1}$.

Lattices

Let P be a poset and $E \subseteq P$.

The infimum, or greatest lower bound, of E in P is $\ell \in P$ such that

(a) If $p \in E$ then $\ell \le p$

(b) If $m \in P$ and m satisfies

if $p \in E$ then $m \le p$

then $m \le \ell$.

The supremum, or least upper bound, of E in P is $r \in P$ such that

(a) If $p \in E$ then $p \le r$

(b) If $t \in P$ and t satisfies

if $p \in E$ then $p \le t$

then $r \le t$.

If $k \in \mathbb{Z}_{>0}$ and $x_1, \ldots, x_k \in P$,
Use the notation $\inf(x_1, \ldots, x_k) = \inf(\{x_1, \ldots, x_k\})$

and $\sup(x_1, \ldots, x_k) = \sup(\{x_1, \ldots, x_k\})$.

A lattice is a poset P such that

if $x, y \in P$ then $\inf(x,y)$ and $\sup(x,y)$

exist in P.

Modular lattices

Let P be a lattice. Use the notation
$$x \wedge y = \inf(x,y) \quad \text{and} \quad x \vee y = \sup(x,y),$$
and the language $x \wedge y$ is "x meet y"
and $x \vee y$ is "x join y"

A modular lattice is a lattice P such that
if $m, n, p \in P$ and $p \le m$ then
$$m \vee (n \wedge p) = (m \vee p) \wedge n.$$

Theorem Let A be a $\mathbb{Z}$-algebra and let V be an A-module.
Let
$$\mathcal{G}(V) = \{A\text{-submodules of } V\}$$
partially ordered by inclusion.
Then $\mathcal{G}(V)$ is a modular lattice.

Proposition Let A be a $\mathbb{Z}$-algebra and let V be an A-module.
Let $M, N, P \in \mathcal{G}(V)$

(a) (infimums exist)

$$\inf(M,N) = M \cap N = \{v \in V \mid v \in M \text{ and } v \in N\}$$

(b) (supremums exist)

$$\sup(M,N) = M+N = \{m+n \mid m \in M \text{ and } n \in N\}$$

(c) (modular law) If $P \subseteq M$ then

$$M + (N \cap P) = (M+N) \cap P$$

(d) (modular property)

$$\frac{M+N}{M} \simeq \frac{N}{M \cap N}.$$

Proposition Let $A = \mathbb{F}_q$ and let V be an A-module so that $V \cong \mathbb{F}_q^n$ where $n = \dim(V)$ as an $\mathbb{F}_q$-vector space. Then

$$G(\mathbb{F}_q^n) = \bigsqcup_{k=0}^{n} G(\mathbb{F}_q^n)_k$$

where $G(\mathbb{F}_q^n)_k = \left\{ \begin{matrix} \mathbb{F}_q\text{-subspaces } W \text{ of } \mathbb{F}_q^n \\ \text{with } \dim(W) = k \end{matrix} \right\}.$

and

$$\mathrm{Card}(G(\mathbb{F}_q^n)_k) = \begin{bmatrix} n \\ k \end{bmatrix}.$$