The binomial theorem and the exponential

For $k \in \mathbb{Z}_{\geq 0}$ define

$$0! = 1 \quad \text{and} \quad k! = k(k-1)\cdots 3\cdot 2\cdot 1 \ \text{ for } k \in \mathbb{Z}_{>0}.$$

For $k \in \{0, 1, \ldots, n\}$ define

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Theorem Let $n \in \mathbb{Z}_{>0}$ and $k \in \{1, \ldots, n\}$.

(a) Let S be a set with $\mathrm{Card}(S) = n$. Then

$\binom{n}{k}$ = number of subsets of S with cardinality k.

(b) $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in $(x+y)^n$

(c) If $k \in \{1, \ldots, n-1\}$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(d) In $\mathbb{C}[x, y]$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$= \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

<u>Corollary</u>

(a) $$f_{(k,\,n-k)} = \binom{n}{k}$$

(b) $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

(c) $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Flath
(see Halverson-Herbig arXiv:0806.3960 page 3)

The ring of <u>formal power series</u> is

$$\mathbb{C}[[x]] = \{ a_0 + a_1 x + a_2 x^2 + \cdots \mid a_i \in \mathbb{C} \}$$

$$\mathbb{C}((x)) = \left\{ a_{-\ell} x^{-\ell} + a_{-\ell+1} x^{-\ell+1} + \cdots \;\middle|\; \begin{array}{l} a_i \in \mathbb{C} \\ \ell \in \mathbb{Z} \end{array} \right\}$$

$$\mathbb{C}[x] = \left\{ a_0 + a_1 x + \cdots \;\middle|\; \begin{array}{l} a_i \in \mathbb{C} \text{ and all} \\ \text{but a fin. no of} \\ a_i \text{ are } 0 \end{array} \right\}$$

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The exponential function

The exponential is

$$\exp(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Theorem

(a) If $xy = yx$ then $\exp(x+y) = \exp(x)\exp(y)$.

(b) $\frac{d}{dx}(\exp(x)) = \exp(x)$.

Theorem

(a) If $p \in \mathbb{C}[[x]]$ and $p(x+y) = p(x)p(y)$

then there exists

$a \in \mathbb{C}$ such that $p(x) = \exp(ax)$

(b) If $p \in \mathbb{C}[[x]]$ and $\frac{dp}{dx} = p$

then there exists $c_0 \in \mathbb{C}$ such that

$p(x) = c_0 \exp(x)$.

The binomial theorem

Define
$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{(a+k-1)!}{(a-1)!}$$
so that
$$n! = (1)_n \quad \text{and} \quad \binom{n}{k} = \frac{(k)_{n-k}}{(1)_k}$$
Define
$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$
Define
$${}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix}; q, z\right] = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{(a_0;q)_k (a_1;q)_k \cdots (a_r;q)_k}{(q;q)_k (b_1;q)_k \cdots (b_r;q)_k} z^k$$
and
$${}_{r+1}F_r\left[\begin{matrix} \alpha_0, \alpha_1, \dots, \alpha_r \\ \beta_1, \dots, \beta_r \end{matrix}; z\right] = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{(\alpha_0)_k (\alpha_1)_k \cdots (\alpha_r)_k}{(1)_k (\beta_1)_k \cdots (\beta_r)_k} z^k$$

<u>Theorem</u> Let $\alpha \in \mathbb{C}$. Then
$$(1-z)^{-\alpha} = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{(\alpha)_k}{k!} z^k = \sum_{k \in \mathbb{Z}_{\geq 0}} \binom{-\alpha}{k} (-z)^k = {}_1F_0[\alpha; z].$$

<u>Proof</u> (One option). Taylor series:
$$\frac{1}{k!} \frac{d^k}{dx^k} (1+x)^\alpha \Big|_{x=0} = \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k!}$$
//.