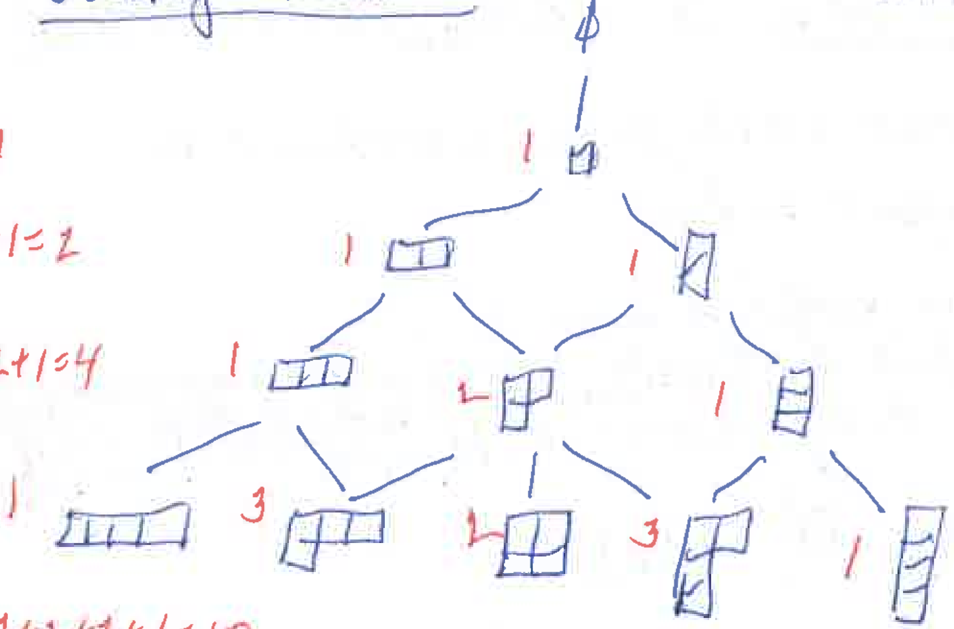
30.10.2024 ①
ADM LI
WILI

Partitions, Bratteli diagrams

Young's lattice (boxes in a corner)



1

$1+1=2$

$1+2+1=4$

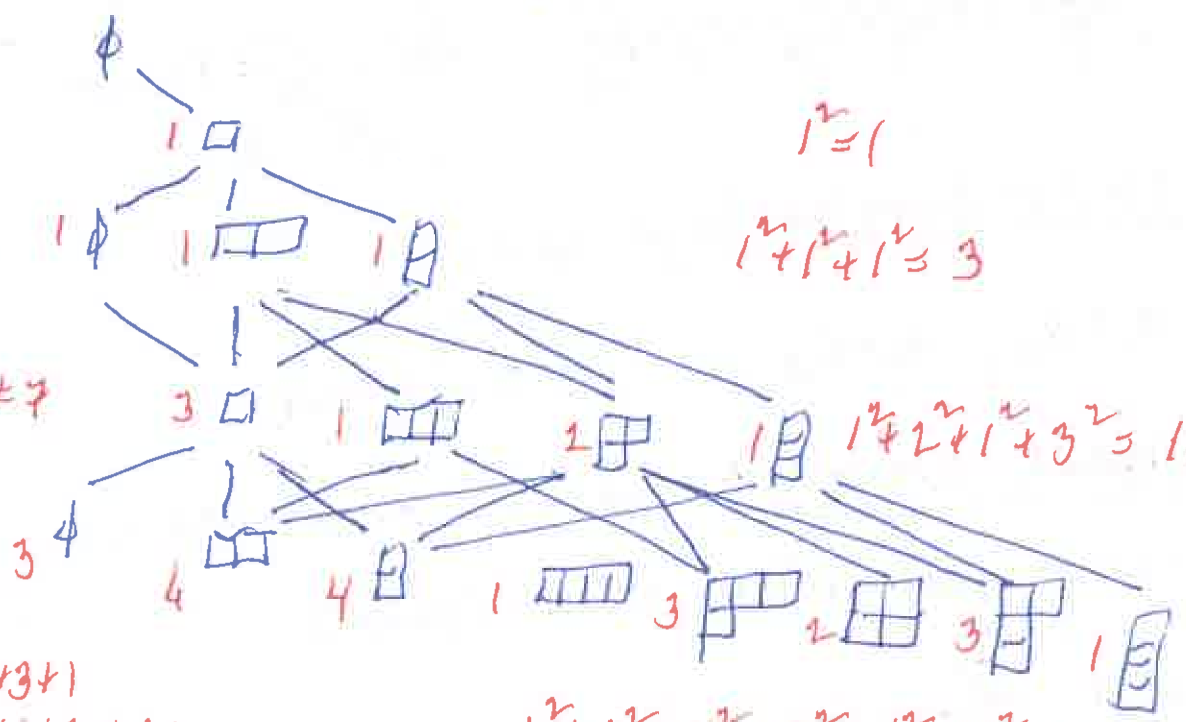
$1+3+2+3+1=10$

$1^2+1^2=2$

$1^2+2^2+1^2=6$

$1^2+3^2+2^2+3^2+1^2=24$

Brauer Bratteli diagram (add and remove)



1

$1+1+1=3$

$3+1+1+1=7$

$1+3+2+3+1$
$+4+6+3=23$

$1^2=1$

$1^2+1^2+1^2=3$

$1^2+2^2+1^2+3^2=15.$

$1^2+3^2+2^2+3^2+1^2+4^2+6^2+3^2=105$

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Temperley-Lieb Bratelli diagram (restrict to two rows)

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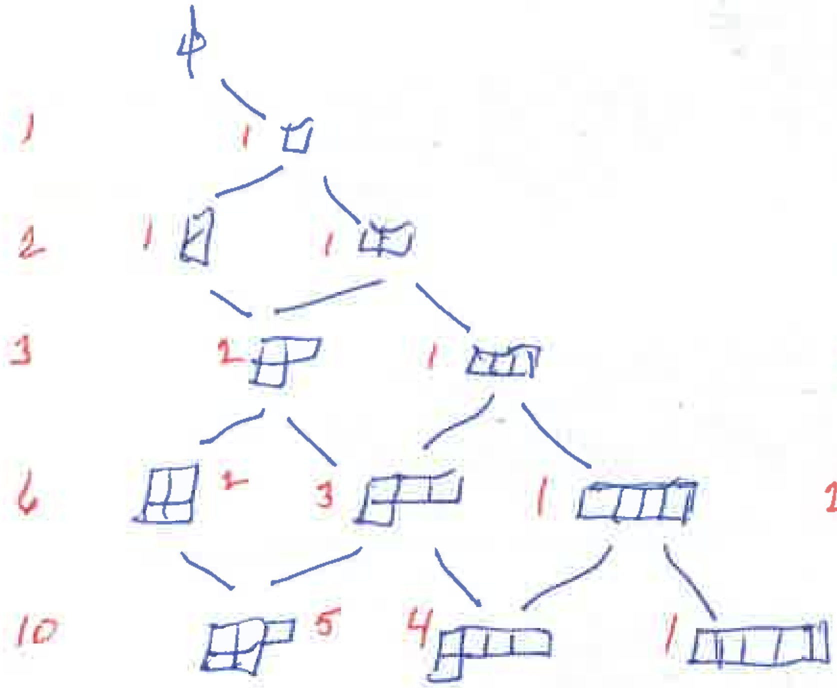


$1^2=1$

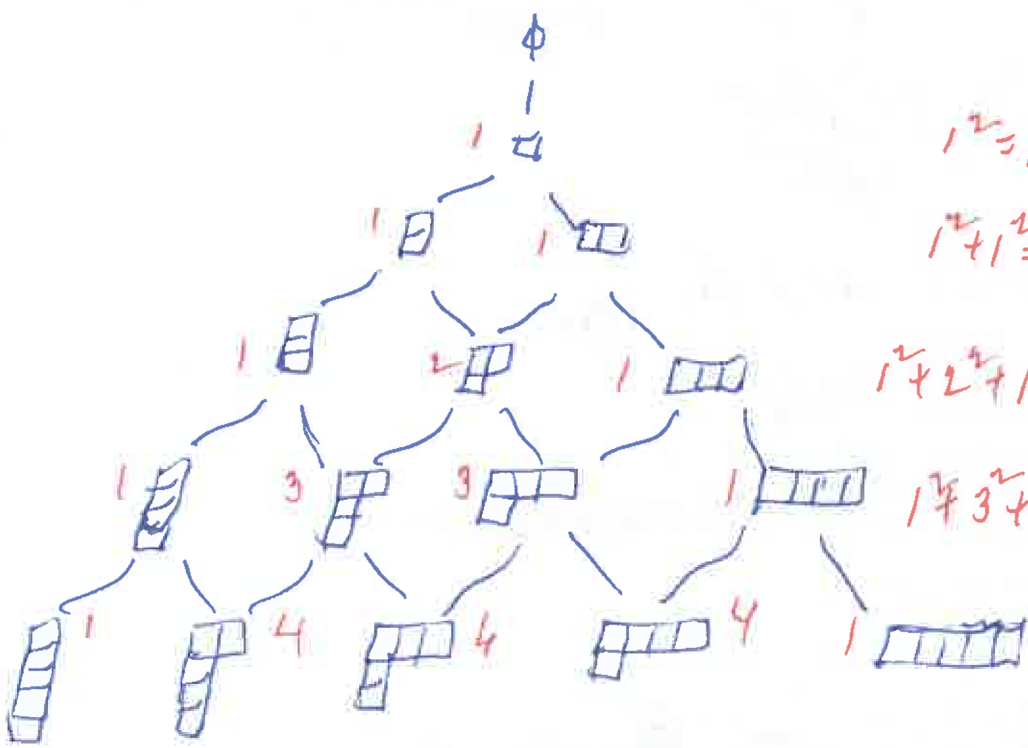
$1^2+1^2=2$

$2^2+1^2=5$

$2^2+3^2+1^2=14$

$5^2+4^2+1^2=42$

Pascal's triangle (restrict to along the wall)



$1^2=1$

$1^2+1^2=2$

$1^2+2^2+1^2=6$

$1^2+3^2+3^2+1^2=20$

$1^2+4^2+6^2+4^2+1^2=70$

30.10.2024

ADH ③

A <u>partition</u> is $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\ell \in \mathbb{Z}_{\geq 0}$, $\lambda_1, \ldots, \lambda_\ell \in \mathbb{Z}_{>0}$ and $\lambda_1 \geq \cdots \geq \lambda_\ell$

A <u>box</u> is an element of $\mathbb{Z}^2$

Identify $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with a set of boxes

$$\lambda = \{(r,c) \mid r \in \{1, \ldots, \ell\} \text{ and } c \in \{1, \ldots, \lambda_r\}\}$$

so that λ has λ_r boxes in row r.

$\lambda = (53311) =$  $= \left\{ \begin{matrix} (1,1), (1,2) \ (1,3) \ (1,4) \ (1,5) \\ (2,1) \ (2,2) \ (2,3) \\ (3,1) \ (3,2) \ (3,3) \\ (4,1) \\ (5,1) \end{matrix} \right\}$

Let

$\ell(\lambda) = \ell$ and $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$,

if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$

Write

$\lambda \subseteq \mu$ if λ is a subset of μ

The <u>conjugate</u> of λ is

$$\lambda' = \{(c,r) \mid (r,c) \in \lambda\}.$$

For $n \in \mathbb{Z}_{\ge 0}$ let

$$\mathcal{Y}_n = \{\text{partitions } \lambda \text{ with } |\lambda| = n\}$$

and

$$\mathcal{Y} = \bigsqcup_{n \in \mathbb{Z}_{\ge 0}} \mathcal{Y}_n$$

Let $\lambda \in \mathcal{Y}_n$.

A <u>standard tableau of shape λ</u> is a function $T : \lambda \to \{1, \dots, n\}$ such that

(a) If $(r,c), (r,c+1) \in \lambda$ then $T(r,c) < T(r,c+1)$

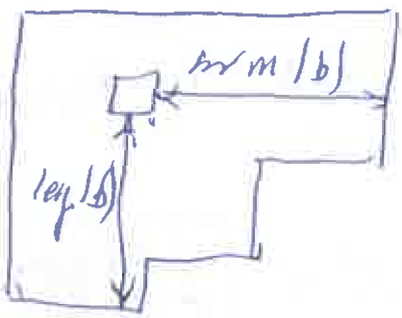
(b) If $(r,c), (r+1,c) \in \lambda$ then $T(r,c) < T(r+1,c)$

Let

$$f_\lambda = \mathrm{Card}\left\{\begin{matrix}\text{standard tableaux } T \\ \text{of shape } \lambda\end{matrix}\right\}$$

If $(r,c) \in \lambda$ define

$$\mathrm{arm}(r,c) = \lambda_r - r$$
$$\mathrm{leg}(r,c) = \lambda'_c - c$$



<u>Theorem</u> Let $\lambda \in \mathcal{Y}_n$

(a) $$f_\lambda = \frac{n!}{\prod_{b \in \lambda} (\mathrm{arm}(b) + \mathrm{leg}(b) + 1)}$$

(b) $$n! = \sum_{\lambda \in \mathcal{Y}_n} f_\lambda^2$$