04.12.2014
ADM ①
L11
W4L2

The crystal of words

$B(\square) = \{ b_1 \xrightarrow{\tilde{f}_1} b_2 \xrightarrow{\tilde{f}_2} \cdots \xrightarrow{\tilde{f}_{n-1}} b_n \}$ with $wt(b_i) = \varepsilon_i$

where $\varepsilon_i = (0, \ldots 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$.

Suppose $n = 3$:

$B = B(\square) = \quad b_1 \xrightarrow{1} b_2 \xrightarrow{2} b_3$

$char(B(\square)) = x_1 + x_2 + x_3$

$$B^{\otimes 2} = B \otimes B = \left\{ \begin{tikzcd} b_1\otimes b_1 \arrow[r, "1"] & b_1\otimes b_2 \arrow[r, "2"] \arrow[d, "1"] & b_1\otimes b_3 \arrow[d, "1"] \\ b_2\otimes b_1 \arrow[d, "2"] & b_2\otimes b_2 \arrow[r, "2"] & b_2\otimes b_3 \arrow[d, "1"] \\ b_3\otimes b_1 \arrow[r, "1"] & b_3\otimes b_2 & b_3\otimes b_3 \end{tikzcd} \right\}$$

$$char(B^{\otimes 2}) = (x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2) + (x_1x_2 + x_1x_3 + x_2x_3)$$

$B^{\otimes 3} = B \otimes B \otimes B = \quad$ 111 112 113

122 123 133

222 223 233

~~223~~ ~~233~~

333

$\begin{matrix}1\\2\\3\end{matrix}$

$\begin{matrix}11\\2\end{matrix}$ $\quad \begin{matrix}11\\3\end{matrix}$

$\begin{matrix}12\\2\end{matrix}$ $\quad \begin{matrix}12\\3\end{matrix}$ $\quad \begin{matrix}13\\2\end{matrix}$ $\quad \begin{matrix}22\\3\end{matrix}$

$\begin{matrix}13\\3\end{matrix}$ $\quad \begin{matrix}23\\3\end{matrix}$

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A *word of length k* in the alphabet $\{b_1, \ldots, b_n\}$ is an element of

$$B(\square)^{\otimes k} = \{ b_{i_1} \otimes \cdots \otimes b_{i_k} \mid i_1, \ldots, i_k \in \{1, \ldots, n\}\}$$

Define

$$\begin{aligned} wt(b_{i_1} \otimes \cdots \otimes b_{i_k}) &= \varepsilon_{i_1} + \cdots + \varepsilon_{i_k} \\ &= \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n, \end{aligned}$$

where $\mu_j = \#(b_j$ in the word $b_{i_1} \otimes \cdots \otimes b_{i_k})$.

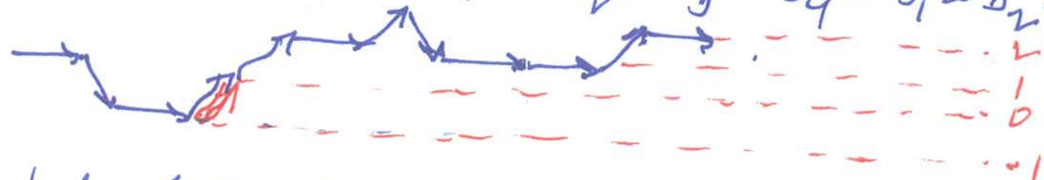
Fix $i \in \{1, \ldots, n-1\}$. Convert the word to a path by setting

$$b_j = \nearrow, \quad b_{j+1} = \searrow, \quad b_r = \longrightarrow \text{ if } r \notin \{j, j+1\}$$

For example, if $i=2$ then

$$b_1 \otimes b_3 \otimes b_4 \otimes b_2 \otimes b_2 \otimes b_1 \otimes b_2 \otimes b_3 \otimes b_4 \otimes b_1 \otimes b_2 \otimes b_4$$

gives



Then let ℓ be the position of the last $\nearrow$ step from the minimum height.

Then

$f_i(b_{i_1} \otimes \cdots \otimes b_{i_k})$ is the same as $b_{i_1} \otimes \cdots \otimes b_{i_k}$ except with b_{i_ℓ} changed from b_j to b_{j+1}.

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<u>theorem</u>

(a) $\mathrm{char}(B(\square)^{\otimes k}) = (x_1 + \cdots + x_n)^k$

(b) Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda_1 + \cdots + \lambda_n = k$. Then

$$(B(\square)^{\otimes k})^+_\lambda \longleftrightarrow \left\{ \begin{matrix} \text{standard tableaux} \\ \text{of shape } \lambda \end{matrix} \right\}$$

$$b_{r(Q(1))} \otimes \cdots \otimes b_{r(Q(n))} \longleftarrow Q$$

(c) If λ does not satisfy $\lambda_1 \geq \cdots \geq \lambda_n$ then $(B(\square)^{\otimes k})^+_\lambda = \emptyset$.

<u>The crystals $B(\lambda)$</u>

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$

A SSYT of shape λ filled from $\{1, \ldots, n\}$ is a function $T : \lambda \to \{1, \ldots, n\}$ such that

(a) if $(r,c), (r,c+1) \in \lambda$ then $T(r,c) \leq T(r,c+1)$

(b) if $(r,c), (r+1,c) \in \lambda$ then $T(r,c) < T(r+1,c)$.

Theorem

(a) There is a unique crystal structure on $B(\lambda)$ such that

$$B(\lambda) \longrightarrow B(\square)^{\otimes k}$$
$$p \longmapsto \text{arabic reading of } p.$$

is a crystal morphism.

(b) $B(\lambda)$ is an irreducible crystal.

Example

$$B\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\right) = \left\{ \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline\end{array} \;\middle|\; \begin{array}{l} i_1, \dots, i_k \in \{1, \dots, n\} \\ i_1 < \dots < i_k \end{array} \right\}$$

with

$$\tilde{f}_j \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline\end{array} = \begin{cases} \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline j+1 \\ \hline \vdots \\ \hline i_k \\ \hline\end{array}, & \text{if } j \in \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline\end{array} \\ 0, & \text{if } j \notin \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline\end{array} \end{cases}$$

and

$$\operatorname{char}(B) = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \cdots x_{i_k} = e_k.$$

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Example $B(\boxed{\square\square\square\square\square}) = \left\{ \boxed{i_1 | \cdots | i_k} \;\middle|\; \begin{matrix} i_1, \ldots, i_k \in \{1, \ldots, n\} \\ i_1 \le \cdots \le i_k \end{matrix} \right\}$

with

$$\hat{f}_j(\boxed{i_1 | \cdots | i_k}) = \begin{cases} \boxed{i_1 \cdots \overbrace{j \cdots j}^{r-1} \; j+1 \cdots i_k} & \text{if } j \text{ appears } r \text{ times in } \boxed{i_1 \; \cdots \; i_k} \\ 0, & \text{if } j \text{ does not appear in } \boxed{i_1 | \cdots | i_k} \end{cases}$$

and

$$\operatorname{char}(B(\boxed{\square\square\square\square})) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \cdots x_{i_k} = h_k .$$

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Crystals for $n=3$

$B = B^{\otimes 1}$ $\quad 1 \xrightarrow{1} 2 \xrightarrow{2} 3$

$B \otimes B = B^{\otimes 2}$

$$\begin{tikzcd}
11 \arrow[r, "1"] & 21 \arrow[r, "1"] \arrow[d, "2"] & 22 \arrow[d, "2"] \\
12 \arrow[dd, "2"] & 31 \arrow[r, "1"] & 32 \arrow[d, "2"] \\
 & & 33 \\
13 \arrow[r, "1"] & 23 &
\end{tikzcd}$$

$B^{\otimes 3}$ 123

$$\begin{tikzcd}
111 \arrow[r, "1"] & 211 \arrow[r, "1"] \arrow[d, "2"] & 221 \arrow[r, "1"] \arrow[d, "2"] & 222 \arrow[d, "2"] \\
 & 311 \arrow[r, "2"] & 321 \arrow[r, "1"] \arrow[d, "2"] & 322 \arrow[d, "2"] \\
 & & 331 \arrow[r, "1"] & 332 \arrow[d, "2"] \\
 & & & 333
\end{tikzcd}$$

$\begin{matrix}1\\2\\3\end{matrix}$ $\quad 123$

$\begin{matrix}1 & 3\\2 & \end{matrix}$

$$\begin{tikzcd}
121 \arrow[r, "1"] \arrow[d, "2"'] & 122 \arrow[r, "2"] & 132 \arrow[r, "2"] & 133 \arrow[d, "1"] \\
131 \arrow[r, "1"] & 231 \arrow[r, "1"] & 232 \arrow[r, "2"] & 233
\end{tikzcd}$$

$\begin{matrix}1 & 2\\3 & \end{matrix}$

$$\begin{tikzcd}
112 \arrow[r, "1"] \arrow[d, "2"] & 212 \arrow[r, "2"] & 312 \arrow[r, "2"] & 313 \arrow[d, "1"] \\
113 \arrow[r, "1"'] & 213 \arrow[r, "1"'] & 223 \arrow[r, "2"'] & 323
\end{tikzcd}$$

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Knuth equivalence

(1) If $x \le y < z$ then $xz \cdot y \sim z \cdot xy$

(2) If $x < y \le z$ then $yz \cdot x \sim y \cdot xz$

Check also Fulton's Young tableaux

need a story
z doesn't want to be near y?

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111	112	113
121 ✓	122	123
131	132	133
211 = ✓	212	213
221	222	223
231	232	233
311	312	313
321	322	333
331	332	333