

4 Generators and relations for S_n and $GL_n(\mathbb{F})$

4.1 A presentation theorem for S_n

Let S_n be the symmetric group of permutation matrices and let

$$s_i = 1 + E_{i,i+1} + E_{i+1,i} - E_{ii} - E_{i+1,i+1}, \quad \text{for } i \in \{1, \dots, n-1\}.$$

The following theorem shows that the symmetric group S_n is a Coxeter group.

Theorem 4.1. *The symmetric group S_n is presented by generators $s_1, \dots, s_{n-1}$ and relations*

$$s_j^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_j s_k = s_k s_j, \quad (\text{Crels})$$

for $j, k \in \{1, \dots, n-1\}$ with $k \notin \{j-1, j+1\}$ and $i \in \{1, \dots, n-2\}$.

Proof sketch. The proof requires four steps:

- (1) Generators A in terms of generators B.
- (2) Generators B in terms of generators A.
- (3) Relations A from relations B.
- (4) Relations B from relations A.

Here

Generators A: { permutation matrices }

Relations A: { matrix multiplication of permutation matrices }

Generators B: { simple transpositions $s_1, \dots, s_{n-1}$ }

Relations B: { the relations in (Crels) }

Further details of the proof are given in Section 4.1.3

□

4.1.1 Length and reduced words

Let $w \in S_n$. A reduced word for w is an expression $w = s_{i_1} \cdots s_{i_\ell}$ with $i_1, \dots, i_\ell \in \{1, \dots, n-1\}$ and ℓ minimal.

The *length* of w is $\ell(w)$, the length of a reduced word for w .

The proof of the following Proposition is given in Section 4.1.2

Proposition 4.2. *Let*

$$\text{Inv}(w) = \{(i, j) \mid i, j \in \{1, \dots, n\} \text{ with } i < j \text{ and } w(i) > w(j)\}. \quad (\text{Invwdef})$$

Then

$$\ell(w) = \text{Card}(\text{Inv}(w)).$$

4.1.2 A reduced word algorithm for $w \in S_n$

Let $w \in S_n$. A *reduced word* for w is an expression $w = s_{i_1} \cdots s_{i_\ell}$ with $i_1, \dots, i_\ell \in \{1, \dots, n-1\}$ and ℓ minimal.

The *length* of w is $\ell(w)$, the length of a reduced word for w .

The following is an explicit algorithm for producing a reduced word for w . It is a version of the ‘row reduction’ algorithm (see Section 4.3.2).

Let $j_1 > 1$ be maximal such that $w_{j_1} \neq 0$. Let

$$w^{(1)} = \begin{cases} w, & \text{if } j_1 \text{ does not exist,} \\ s_1 \cdots s_{j_1-1} w, & \text{if } j_1 \text{ exists.} \end{cases}$$

Let $j_2 > 2$ be maximal such that $w_{j_2}^{(1)} \neq 0$. If j_2 does not exist set $w^{(2)} = w^{(1)}$ and if j_2 does exist set

$$w^{(2)} = \begin{cases} w^{(1)}, & \text{if } j_2 \text{ does not exist,} \\ s_2 \cdots s_{j_2-1} w^{(1)}, & \text{if } j_2 \text{ exists.} \end{cases}$$

Continue this process to produce $w^{(1)}, \dots, w^{(n)}$. Then $w^{(n)} = 1$ and

$$w = \cdots (s_{j_2-1} \cdots s_2)(s_{j_1-1} \cdots s_1) \quad \text{is a reduced word for } w. \quad (\text{gdyredwd})$$

Let us state this structural result as a theorem.

Theorem 4.3. *Let $w \in W$ and define*

$$\text{Inv}(w) = \{(i, j) \mid i, j \in \{1, \dots, n\} \text{ with } i < j \text{ and } w(i) > w(j)\}.$$

For $i \in \{1, \dots, n-1\}$ let j_i be given by $j_i - i = \#\{(i, k) \in \text{Inv}(w) \mid k \in \{i+1, \dots, n\}\}$. Then

$$w = \cdots (s_{j_2-1} \cdots s_2)(s_{j_1-1} \cdots s_1) \quad \text{is a reduced word for } w.$$

and

$$\ell(w) = \text{Card}(\text{Inv}(w)).$$

Example 4.1. *If $w \in S_4$ is given by*

$$w = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{then} \quad s_3(s_2s_3)(s_1s_2w) = s_3(s_2s_3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = s_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 1,$$

so that $w = (s_2s_1)(s_3s_2)s_3$. □

The *longest element* of S_n is the permutation w_0 given by $w_0(i) = n - i$ for $i \in \{1, \dots, n\}$. If $w \in S_n$ then the greedy reduced word for w is a subword of the reduced word of the longest element given by

$$(s_{n-1} \cdots s_2s_1)(s_{n-1} \cdots s_3s_2) \cdots (s_{n-1}s_{n-2})s_{n-1} = w_0. \quad (\text{longelt})$$

4.1.3 Proof of the presentation theorem for S_n

The *simple transpositions* in S_n are the matrices $s_i = s_{i,i+1}$,

$$s_i = \begin{matrix} & & & i & i+1 & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ i & & & & & & \\ i+1 & & & & & & \end{matrix} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 1 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}, \quad \text{for } i \in \{1, \dots, n-1\}. \quad (4.1)$$

Proposition 4.4. *The symmetric group S_n is presented by generators $s_1, s_2, \dots, s_{n-1}$ and relations*

$$s_i^2 = 1 \quad \text{and} \quad s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1} \quad \text{and} \quad s_k s_\ell = s_\ell s_k, \quad (4.2)$$

for $i, j, k, \ell \in \{1, \dots, n-1\}$ with $j \neq n-1$ and $k \neq \ell \pm 1$.

Proof.

Generators A: the set of permutation matrices.

Relations A: all products of permutations $w_1 w_2$ given by matrix multiplication.

Generators B: $s_1, \dots, s_{n-1}$.

Relations B: As given in (4.2).

The proof is accomplished in four steps:

- (1) Write generators B in terms of generators A.
- (2) Deduce relations B from relations A.
- (3) Write generators A in terms of generators B.
- (4) Deduce relations A from relations B.

Step 1: Generators B in terms of generators A. This is provided by (4.1).

Step 2: Relations B from relations A. This step is given the following matrix computations:

$$s_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$s_1 s_2 s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$s_2 s_1 s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so that $s_1 s_2 s_1 = s_2 s_1 s_2$ and

$$s_1 s_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$s_3 s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

so that $s_1 s_3 = s_3 s_1$.

Step 3: Generators A in terms of generators B. This step is provided by gdyredwd.

Step 4: Relations A from relations B.

$$\begin{aligned} s_i(s_{j-1} \cdots s_2 s_1) &= s_{j-1} \cdots s_{i+2} s_i s_{i+1} s_i s_{i-1} \cdots s_2 s_1, && \text{by the third set of relations in (4.2),} \\ &= s_{j-1} \cdots s_{i+2} s_{i+1} s_i s_{i+1} s_{i-1} \cdots s_2 s_1, && \text{by the second set of relations in (4.2),} \\ &= (s_{j-1} \cdots s_{i+2} s_{i+1} s_i s_{i-1} \cdots s_2 s_1) s_i, && \text{by the third set of relations in (4.2),} \end{aligned}$$

So $s_i w$ can be written in normal form. By Step 3, w_1 can be written as a product of simple transpositions, so one simple transposition at a time, $w_1 w$ can be written in normal form. $\square$

4.1.4 The graph of reduced words for $w \in S_n$

Define a graph $\Gamma(w)$ with

$$\begin{aligned} \text{Vertices: } & \{\text{reduced words of } w\} \\ \text{Edges: } & u \rightarrow u' \text{ if } u' = s_{i_1} \cdots s_{i_\ell} \text{ is obtained from } u = s_{j_1} \cdots s_{j_\ell} \text{ by applying} \\ & \text{a relation } s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ or a relation } s_i s_j = s_j s_i \text{ with } j \notin \{i-1, i+1\}. \end{aligned}$$

Theorem 4.5. *Let $w \in S_n$. The graph $\Gamma(w)$ of reduced words of w is connected.*

Proof. Let

$$w = s_{i_1} \cdots s_{i_\ell} \quad \text{and} \quad w = s_{j_1} \cdots s_{j_\ell}$$

be reduced words.

Case 1: $i_1 = j_1$. The two reduced words for w have the same first letter. By induction, the reduced words $v = s_{i_2} \cdots s_{i_\ell}$ and $v = s_{j_2} \cdots s_{j_\ell}$ are connected.

Case 2: $i_1 \neq j_1$. Since $\ell(s_{j_1} w) < \ell(w)$ then there exists k such that $s_{j_1} w = s_{i_1} \cdots s_{i_{k-1}} \cancel{s_{i_k}} s_{i_{k+1}} \cdots s_{i_\ell}$.

Case 2a: $k \neq \ell$. Then

$$\begin{aligned} w &= s_{j_1} \cdots s_{j_\ell} \\ w &= s_{j_1} s_{i_1} \cdots s_{i_{k-1}} \cancel{s_{i_k}} s_{i_{k+1}} \cdots s_{i_\ell} \quad \text{and} \\ w &= s_{i_1} \cdots s_{i_\ell} \end{aligned}$$

are all reduced words for w . Since the first factor is the same in the first two of these they are connected. Since the last factor is the same in the last two of these they are connected. So, by transitivity, the first is connected to the last.

Case 2b: $k = \ell$ and $j_1 \notin \{i_1 - 1, i_1 + 1\}$. Then

$$\begin{aligned} w &= s_{j_1} \cdots s_{j_\ell}, \\ w &= s_{j_1} s_{i_1} \cdots s_{i_{\ell-1}}, \\ w &= s_{i_1} s_{j_1} \cdots s_{i_{\ell-1}} \quad \text{and} \\ w &= s_{i_1} s_{i_2} \cdots s_{i_\ell} \end{aligned}$$

and the first two are connected since they have the same first letter, the middle two are connected by the move $s_{j_1}s_{i_1} = s_{j_1}s_{i_1}$ and the last two are connected since they have the same first letter.

Case 2c: $k = \ell$ and $j_1 \in \{i_1 - 1, i_1 + 1\}$. Then

$$\begin{aligned} w &= s_{i_1}s_{i_2} \cdots s_{i_\ell}, \\ w &= s_{i_1}s_{j_1}s_{i_1} \cdots s_{i_{r-1}}\cancel{s_{i_r}}s_{i_{r+1}} \cdots s_{i_{\ell-1}}, \\ w &= s_{j_1}s_{i_1}s_{j_1} \cdots s_{i_{r-1}}\cancel{s_{i_r}}s_{i_{r+1}} \cdots s_{i_{\ell-1}}, \quad \text{and} \\ w &= s_{j_1}s_{j_2} \cdots s_{j_\ell}, \end{aligned}$$

and the first two are connected since they have the same first letter, the middle two are connected by the move $s_{i_1}s_{j_1}s_{i_1} = s_{j_1}s_{i_1}s_{j_1}$ and the last two are connected since they have the same first letter. $\square$

4.2 Introducing root system notation

Positive roots label reflections. For $i \in \{1, \dots, n\}$, let

$$\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n \text{ with 1 in the } i\text{th spot and all other entries 0.}$$

Then

$$\{\varepsilon_1, \dots, \varepsilon_n\} \quad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}^n.$$

The *positive roots* are the elements of

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \dots, n\} \text{ such that } i < j\}. \quad (\text{RpdefSn})$$

and the *transpositions*, or *reflections*, in S_n are

$$s_{ij} = 1 + E_{ij} + E_{ji} - E_{ii} - E_{jj}, \quad \text{for } i, j \in \{1, \dots, n\} \text{ with } i \neq j.$$

Simple roots label simple transpositions. The *simple roots* are $\alpha_1, \dots, \alpha_{n-1}$ given by

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \dots, \quad \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n.$$

The *simple transpositions*, or *simple reflections* are

$$s_1 = s_{12}, \quad s_2 = s_{23}, \quad \dots, \quad s_{n-1} = s_{n-1,n}.$$

Inversion sets in terms of roots. Let $w \in S_n$. Recall from (Invwdef) that $\text{Inv}(w)$ is defined by

$$\text{Inv}(w) = \{(i, j) \mid i, j \in \{1, \dots, n\} \text{ with } i < j \text{ and } w(i) > w(j)\}.$$

Using the action of S_n on $\mathbb{Z}^n$ that permutes the coordinates and identifying $\varepsilon_i - \varepsilon_j \in R^+$ with the pair (i, j) write

$$\text{Inv}(w) = \{\alpha \in R^+ \mid w\alpha \notin R^+\}.$$

The longest element w_0 of S_n is given by $w_0(i) = n - i$ for $i \in \{1, \dots, n\}$ and has

$$\text{Inv}(w_0) = R^+.$$

Proposition 4.6. *Let $w \in S_n$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w . The root sequence for the reduced word $w^{-1} = s_{i_\ell} \cdots s_{i_1}$ is $(\beta_1, \dots, \beta_\ell)$ given by*

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}\alpha_{i_2}, \quad \dots \quad \beta_\ell = s_{i_1}s_{i_2} \cdots s_{i_{\ell-1}}\alpha_{i_\ell}.$$

Then

$$\{\beta_1, \dots, \beta_\ell\} = \text{Inv}(w^{-1}).$$

Proof. The proof is by induction on ℓ , where ℓ is the length of the reduced word for w . If $j \in \{1, \dots, n\}$ and $\ell(s_j w) > \ell(w)$ then $w^{-1}s_j = s_{i_1} \cdots s_{i_\ell}s_j$ is a reduced word for $w^{-1}s_j$ and the root sequence for $w^{-1}s_j$ is

$$\gamma_1 = \alpha_j, \quad \gamma_2 = s_j \alpha_{i_1}, \quad \gamma_3 = s_j s_{i_1} \alpha_{i_2}, \quad \dots, \quad \gamma_{\ell+1} = s_j s_{i_1} \cdots s_{i_{\ell-1}} \alpha_{i_\ell}.$$

So

$$\{\gamma_1, \dots, \gamma_{\ell+1}\} = \{\alpha_j\} \cup s_j \text{Inv}(w^{-1}) = \text{Inv}(w^{-1}s_j),$$

where the last equality follows from $\text{Inv}(w^{-1}) = \{(i, j) \mid i < j \text{ and } w^{-1}(i) > w^{-1}(j)\}$ and that

$$w^{-1}s_j = (w^{-1}(1), \dots, w^{-1}(j+1), w^{-1}(j), \dots, w^{-1}(n)), \quad \text{in one line notation.}$$

(in other words, in one line notation, the permutations w^{-1} and $w^{-1}s_j$ differ only in ordering of the j th and the $(j+1)$ st entries). $\square$

4.3 A presentation theorem for $GL_n(\mathbb{F})$

Let $\mathbb{F}$ be a field, let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices with entries in $\mathbb{F}$.

- An $n \times n$ *invertible matrix* is an $n \times n$ matrix $A \in M_n(\mathbb{F})$ such that

$$\text{there exists } A^{-1} \in M_n(\mathbb{F}) \quad \text{such that} \quad A^{-1}A = 1 \text{ and } AA^{-1} = 1.$$

- The *general linear group* is

$$GL_n(\mathbb{F}) = \{n \times n \text{ invertible matrices with entries in } \mathbb{F}\}.$$

The invertible elements of the field $\mathbb{F}$ are the elements of

$$\mathbb{F}^\times = \{d \in \mathbb{F} \mid d \neq 0\} = \{1 \times 1 \text{ invertible matrices with entries in } \mathbb{F}\} = GL_1(\mathbb{F}).$$

Theorem 4.7. *The group $GL_n(\mathbb{F})$ is presented by generators*

$$y_i(c), \quad h_j(d), \quad x_{k\ell}(c), \quad \text{for} \quad \begin{array}{l} c \in \mathbb{F}, d_1, \dots, d_n \in \mathbb{F}^\times, \\ i \in \{1, \dots, n-1\}, j \in \{1, \dots, n\} \\ k, \ell \in \{1, \dots, n\} \text{ with } k < \ell. \end{array} \quad (\text{GensB})$$

with the following relations:

- The reflection relation is

$$y_i(c_1)y_i(c_2) = \begin{cases} y_i(c_1 + c_2^{-1})h_i(c_2)h_{i+1}(-c_2^{-1})x_{i,i+1}(c_2^{-1}), & \text{if } c_2 \neq 0, \\ x_{i,i+1}(c_1), & \text{if } c_2 = 0. \end{cases} \quad (\text{refrel})$$

- The building relations are

$$\begin{aligned} y_i(c_1)y_{i+1}(c_2)y_i(c_3) &= y_{i+1}(c_3)y_i(c_1c_3 + c_2)y_{i+1}(c_1), \\ y_i(c_1)y_j(c_2) &= y_j(c_2)y_i(c_1), \quad \text{if } j \notin \{i-1, i+1\}. \end{aligned} \quad (\text{bldrel})$$

- The x -interchange relations are

$$\begin{aligned} x_{ij}(c_1)x_{ij}(c_2) &= x_{ij}(c_1 + c_2), \\ x_{ij}(c_1)x_{ik}(c_2) &= x_{ik}(c_2)x_{ij}(c_1), & x_{ik}(c_1)x_{jk}(c_2) &= x_{jk}(c_2)x_{ik}(c_1), & (\text{xint}) \\ x_{ij}(c_1)x_{jk}(c_2) &= x_{jk}(c_2)x_{ij}(c_1)x_{ik}(c_1c_2), & x_{jk}(c_1)x_{ij}(c_2) &= x_{ij}(c_2)x_{jk}(c_1)x_{ik}(-c_1c_2), \end{aligned}$$

where $i < j < k$.

- The h -processing relations are

$$h_i(d_1)h_j(d_2) = h_j(d_2)h_i(d_1) \quad \text{and} \quad h_i(d_1)h_i(d_2) = h_i(d_1d_2), \quad (\text{hhrel})$$

- Letting $h(d_1, \dots, d_n) = h_1(d_1) \cdots h_n(d_n)$, the h -past- y relation is

$$h(d_1, \dots, d_n)y_i(c) = y_i(cd_id_{i+1}^{-1})h(d_1, \dots, d_{i-1}, d_{i+1}, d_i, d_{i+2}, \dots, d_n). \quad (\text{hpyrel})$$

- Letting $h(d_1, \dots, d_n) = h_1(d_1) \cdots h_n(d_n)$, the h -past- x relation is

$$h(d_1, \dots, d_n)x_{ij}(c) = x_{ij}(cd_id_j^{-1})h(d_1, \dots, d_n). \quad (\text{hpxrel})$$

- The x -past- y relations are

$$\begin{aligned} x_{i,i+1}(c_1)y_i(c_2) &= y_i(c_1 + c_2)x_{i,i+1}(0), \\ x_{ik}(c_1)y_k(c_2) &= y_k(c_2)x_{ik}(c_1c_2)x_{i,k+1}(c_1), \quad x_{i,k+1}(c_1)y_k(c_2) = y_k(c_2)x_{ik}(c_1), \quad (\text{xpyrel}) \\ x_{ij}(c_1)y_i(c_2) &= y_i(c_2)x_{i+1,j}(c_1), \quad x_{i+1,j}(c_1)y_i(c_2) = y_i(c_2)x_{ij}(c_1)x_{i+1,j}(-c_1c_2), \end{aligned}$$

where $i < k$ and $i + 1 < j$.

Proof sketch.

Generators A: { invertible matrices }

Relations A: { matrix multiplication of invertible matrices }

Generators B: { row reducers $y_i(c)$, diagonal generators $h_i(d)$, and elementary matrices $x_{ij}(c)$ }

Relations B: { the interchange relations in the statement }

The proof requires four steps:

- (1) Generators A in terms of generators B.
- (2) Generators B in terms of generators A.
- (3) Relations A from relations B.
- (4) Relations B from relations A.

Step(2), which requires the expression of the Generators B in terms of the generators A, is provided by the definitions in section [4.3.1](#).

Step (4), which derives the Relations B from relations A (matrix multiplication), is checked in section [4.3.3](#).

Step (1), which describes how to write an invertible matrix in terms of the elementary matrices is given in section [4.3.2](#).

Step (3), which describes how to derive Relations A (matrix multiplication) from the relations B, is checked in section [4.3.4](#).

□

4.3.1 Elementary matrices, diagonal generators and row reducers

Let

E_{ij} be the $n \times n$ matrix with 1 in the (i, j) entry and 0 elsewhere.

- The elementary matrices in $GL_n(\mathbb{F})$ are the matrices

$$x_{ij}(c) = 1 + cE_{ij}, \quad \text{for } i, j \in \{1, \dots, n\} \text{ with } i \neq j \text{ and } c \in \mathbb{F},$$

- The diagonal generators in $GL_n(\mathbb{F})$ are the matrices

$$h_i(d) = 1 + (d - 1)E_{ii}, \quad \text{for } i \in \{1, \dots, n\} \text{ and } d \in GL_1(\mathbb{F}).$$

- The row reducers in $GL_n(\mathbb{F})$ are

$$y_i(c) = 1 + (c - 1)E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} \quad \text{for } i \in \{1, \dots, n-1\} \text{ and } c \in \mathbb{F}.$$

$$y_i(c) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \quad \text{and} \quad y_i(c)^{-1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & 1 & \\ & & & 1 & -c & \\ & & & & & 1 \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

4.3.2 A reduced word algorithm for $g \in GL_n(\mathbb{F})$

Let $g \in GL_n(\mathbb{F})$ so that g is an $n \times n$ invertible matrix.

The following is an explicit algorithm for writing g as a product of row reducers $y_i(c)$, diagonal generators $h_i(d)$ and upper triangular elementary matrices $x_{ij}(a_{ij})$. This procedure is no different than the usual row reduction procedure: namely, a way of writing an invertible matrix g in a ‘normal form’ as a product of elementary matrices by the ‘row reduction’ algorithm. This process is a generalization of the algorithm for construction the greedy reduced word for a permutation w that was introduced in Section [4.1.2](#).

Let $j_1 > 1$ be maximal such that $g(j_1, 1) \neq 0$. Let If $j_1 = 1$ then let $g^{(1)} = g$. If $j_1 \neq 1$ then let

$$g^{(1)} = y_1 \left(\frac{g(1,1)}{g(j_1,1)} \right)^{-1} y_2 \left(\frac{g(1,2)}{g(j_1,1)} \right)^{-1} \cdots y_{j_1-1} \left(\frac{g(j_1-1,1)}{g(j_1,1)} \right)^{-1} g.$$

Let $j_2 > 2$ be maximal such that $g^{(1)}(j_2, 2) \neq 0$. If $j_2 = 2$ then let $g^{(2)} = g^{(1)}$. If $j_2 \neq 2$ then let

$$g^{(2)} = y_2 \left(\frac{g^{(1)}(2,2)}{g^{(1)}(j_2,2)} \right)^{-1} y_3 \left(\frac{g^{(1)}(3,2)}{g^{(1)}(j_2,2)} \right)^{-1} \cdots y_{j_2-1} \left(\frac{g^{(1)}(j_2-1,2)}{g^{(1)}(j_2,2)} \right)^{-1} g^{(1)}.$$

Continuing this process will produce $g^{(n)}$ which has the property that

the first nonzero entry in row $j+1$ is to the right of the first nonzero entry in row j .

Since g is invertible then $g^{(n)}$ must be upper triangular.

Let $b = g^{(n)}$. Then

$$g = \cdots (y_{j_2-1} \left(\frac{g^{(1)}(j_2-1,2)}{g^{(1)}(j_2,2)} \right) \cdots y_3 \left(\frac{g^{(1)}(3,2)}{g^{(1)}(j_1,2)} \right) y_2 \left(\frac{g^{(1)}(2,2)}{g^{(1)}(j_2,2)} \right)) \cdot (y_{j_1-1} \left(\frac{g(j_1-1,1)}{g(j_1,1)} \right) \cdots y_2 \left(\frac{g(2,1)}{g(j_1,1)} \right) y_1 \left(\frac{g(1,1)}{g(j_1,1)} \right)) \cdot b \quad (\text{GLfact})$$

Let us state this structural result as a theorem. In stating Theorem [4.8](#) it is useful to take advantage of the corresponding structural result for a permutation w in the symmetric group, namely the box greedy reduced word for w that is produced in Section [4.1.2](#).

Theorem 4.8. *Let B be the group of upper triangular invertible matrices. Let $g \in GL_n(\mathbb{C})$. There there exists a unique $w \in S_n$ and unique $c_1, \dots, c_\ell \in \mathbb{F}$ a unique $b \in B$ such that*

$$g = y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) b,$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is the greedy reduced word for w .

Example 4.2. *Let*

$$g = \begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Since

$$\begin{aligned} g &= \begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2\left(\frac{1}{8}\right) \begin{pmatrix} 7 & 6 & 2 & 4 \\ 8 & 6 & 3 & 5 \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix} \\ &= y_2\left(\frac{1}{8}\right)y_1\left(\frac{7}{8}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & \frac{1}{4} & -\frac{3}{8} \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix} \\ &= y_2\left(\frac{1}{8}\right)y_1\left(\frac{7}{8}\right)y_3\left(\frac{29}{4}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & \frac{1}{4} & -\frac{3}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix} \\ &= y_2\left(\frac{1}{8}\right)y_1\left(\frac{7}{8}\right)y_3\left(\frac{29}{4}\right)y_2\left(\frac{3}{4}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{1}{2} & -\frac{15}{8} \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix} \\ &= y_2\left(\frac{1}{8}\right)y_1\left(\frac{7}{8}\right)y_3\left(\frac{29}{4}\right)y_2\left(\frac{3}{4}\right)y_3\left(\frac{4}{5}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \\ 0 & 0 & 0 & -\frac{71}{40} \end{pmatrix}. \end{aligned}$$

□

The examples

$$x_{34}(c_{34})x_{24}(c_{24})x_{14}(c_{14})x_{23}(c_{23})x_{13}(c_{13})x_{12}(c_{12}) = \begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & 1 & c_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} &h_1(d_1)h_2(d_2)h_3(d_3)h_4(d_4) \cdot x_{34}(c_{34})x_{24}(c_{24})x_{14}(c_{14})x_{23}(c_{23})x_{13}(c_{13})x_{12}(c_{12}) \\ &= h_1(d_1)h_2(d_2)h_3(d_3)h_4(d_4) \begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & 1 & c_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d_1 & c_{12} & c_{13} & c_{14} \\ 0 & d_2 & c_{23} & c_{24} \\ 0 & 0 & d_3 & c_{34} \\ 0 & 0 & 0 & d_4 \end{pmatrix} \end{aligned}$$

show how an upper triangular matrix is written in normal form as a product of $h_i(d)$ and $x_{jk}(c)$.

For $i, j \in \{1, \dots, n\}$ with $i < j$ let $a_{ij} \in \mathbb{F}$. The product

$$\left(\prod_{i < j} x_{ij}(a_{ij}) \right) \quad \text{is in matrix parametrization order if}$$

$x_{jk}(a_{jk})$ appears before $x_{ik}(a_{ik})$ for $j > i$, and $x_{j\ell}(a_{j\ell})$ appears before $x_{ik}(a_{ik})$ for $\ell > k$.

Theorem 4.9. *Let $g \in GL_n(\mathbb{C})$. There there exists a unique $w \in S_n$ and unique $c_1, \dots, c_\ell \in \mathbb{F}$ and unique $d_1, \dots, d_n \in \mathbb{F}^\times$ and unique $a_{ij} \in \mathbb{F}$ for $i, j \in R^+$ such that*

$$g = y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) \cdot h_1(d_1) \cdots h_n(d_n) \cdot \left(\prod_{i < j} x_{ij}(a_{ij}) \right),$$

where $w = s_{i_1} \cdots s_{i_\ell}$ be the greedy reduced word for w and the product $\left(\prod_{i < j} x_{ij}(a_{ij}) \right)$ is in matrix parametrization order.

Example 4.3. If $g = \begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ as in Example 4.2 then

$$g = y_2(\frac{1}{8})y_1(\frac{7}{8})y_3(\frac{29}{4})y_2(\frac{3}{4})y_3(\frac{4}{5})h_1(8)h_2(1)h_3(-\frac{5}{8})h_4(-\frac{71}{40})x_{34}(-\frac{49}{8})x_{24}(2)x_{14}(5)x_{23}(1)x_{13}(3)x_{12}(6)$$

is an expression for g purely in terms of the row reducers, the diagonal geberators and the upper triangular elementary matrices. $\square$

4.3.3 Obtaining the interchange relations from matrix multiplication

Proof of the reflection relation:

If $c_1 \neq 0$ and $c_2 \neq 0$ then

$$\begin{aligned} y_1(c_1)y_1(c_2) &= \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1 c_2 + 1 & c_1 \\ c_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1 + c_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 0 & -c_2^{-1} \end{pmatrix} = \begin{pmatrix} c_1 + c_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 0 \\ 0 & -c_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & c_2^{-1} \\ 0 & 1 \end{pmatrix} \\ &= y_1(c_1 + c_2^{-1})h_1(c_2)h_2(-c_2^{-1})x_{12}(c_2^{-1}). \end{aligned}$$

If $c_2 = 0$ then

$$y_1(c_1)y_1(0) = \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} = x_{12}(c_1).$$

Proof of the building relation:

$$\begin{aligned} \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} c_1 c_3 + c_2 & 1 & 0 \\ c_3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 c_3 + c_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

The computation for the proof of the first x -interchange relation is:

$$\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_1 + c_2 \\ 0 & 1 \end{pmatrix}$$

The key computation for the proof of the h -past- y relation is:

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} cd_1 & d_1 \\ d_2 & 0 \end{pmatrix} = \begin{pmatrix} cd_1 d_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_2 & 0 \\ 0 & d_1 \end{pmatrix}$$

Key computations for the proof of the x -past- y relations are:

$$\begin{aligned} \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} c_1 + c_2 & 1 \\ 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & c_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & c_1 c_2 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c_1 c_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & c_1 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} c_2 & 1 & c_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & c_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c_1 c_2 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The remaining relations are derived similarly.

4.3.4 Deriving matrix multiplication from the interchange relations

Suppose that g_1 and g_2 are two expressions given in the normal form of Theorem [4.9](#). The goal is to use the Relations B to rearrange and simplify the product $g_1 g_2$.

Step 1. The h -past- x relations and h -past- y relations and the x -past- y relations allow us to move all the row reducers $y_i(c)$ to the left, all the elementary matrices $x_{ij}(c)$ to the right so that all the diagonal generators $h_i(d)$ are in the middle.

Step 2. The hh -relations allow us to write the product of the diagonal generators in the form $h(d_1, \dots, d_n)$.

Step 3. The reflection relation and the building relation allow us to reduce the y -product to being a y -product for a reduced word of a permutation. Then the theorem that the graph of reduced words is conneted allows us to arrange this reduced word to be the greedy reduced word for w .

Step 4. The x -interchange relations allow us to put the $x_{ij}(c)$ into its appropriate place in the matrix presentation order.

In combination these moves rearrange the product $g_1 g_2$ into normal form.

4.4 Introducing root system notation

Roots label elementary matrices. For $i \in \{1, \dots, n\}$, let

$$\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n \text{ with 1 in the } i\text{th spot and all other entries 0.}$$

Then

$$\{\varepsilon_1, \dots, \varepsilon_n\} \quad \text{is a } \mathbb{Z}\text{-basis of } \mathbb{Z}^n.$$

Use the notation

$$x_{\varepsilon_i - \varepsilon_j}(c) = x_{ij}(c), \quad \text{for } i, j \in \{1, \dots, n\} \text{ with } i \neq j.$$

The set of *roots* is

$$R = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \dots, n\} \text{ and } i \neq j\}$$

and

the elementary matrices are the $x_\alpha(c)$, for $\alpha \in R$ and $c \in \mathbb{F}$.

The positive roots label the elementary matrices in B . Let B be the group of upper triangular invertible matrices. The *positive roots* are the elements of

$$\begin{aligned} R^+ &= \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \dots, n\} \text{ and } i \neq j \text{ and } x_{ij}(c) \in B\} \\ &= \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \dots, n\} \text{ such that } i < j\}. \end{aligned} \quad (\text{Rpdef})$$

The group of upper triangular matrices with 1 on the diagonal (unipotent upper triangular matrices) is

$$U = \left\{ \left(\prod_{i < j} x_{ij}(a_{ij}) \right) \mid a_{ij} \in \mathbb{F} \right\} \quad \text{and} \quad U \text{ is generated by } \{x_{12}(c), x_{23}(c), \dots, x_{n-1,n}(c) \mid c \in \mathbb{F}\}.$$

The *simple roots* are $\alpha_1, \dots, \alpha_{n-1}$ given by

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \dots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n.$$

For $\alpha \in R^+$, let

$$\mathcal{X}_\alpha = \{x_\alpha(c) \mid c \in \mathbb{F}\}.$$

Then

$$U = \prod_{\alpha \in R^+} \mathcal{X}_\alpha \quad \text{and} \quad U \text{ is generated by } \{\mathcal{X}_{\alpha_1}, \dots, \mathcal{X}_{\alpha_n}\}.$$

5 Bruhat decomposition and the Hecke algebra

5.1 Reviewing the normal form theorem

The normal form theorem, Theorem 4.9 is the following.

Theorem 5.1. *Let $g \in GL_n(\mathbb{C})$. There there exists a unique $w \in S_n$ and unique $c_1, \dots, c_\ell \in \mathbb{F}$ and unique $d_1, \dots, d_n \in \mathbb{F}^\times$ and unique $a_{ij} \in \mathbb{F}$ for $i, j \in \{1, \dots, n\}$ with $i < j$, such that*

$$g = y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) \cdot h_1(d_1) \cdots h_n(d_n) \cdot \left(\prod_{i < j} x_{ij}(a_{ij}) \right),$$

where $w = s_{i_1} \cdots s_{i_\ell}$ be the greedy reduced word for w and the product $\left(\prod_{i < j} x_{ij}(a_{ij}) \right)$ is in matrix parametrization order.

This theorem has such powerful consequences that it is useful to view it state it in several slightly different (slightly more general) incarnations. First, use the relations (bldrel) and Theorem 4.5 to generalize the statement from the box greedy reduced word to any chosen reduced word and use the relations (xint) to change from the matrix parametrization order to any chosen order of the $x_{ij}(a)$.

Theorem 5.2. *Suppose that for each $w \in S_n$ a fixed reduced word for w has been chosen. Also*

choose a specific order on $\{\text{pairs } (i, j) \text{ with } i, j \in \{1, \dots, n\} \text{ and } i < j\}$.

Let $g \in GL_n(\mathbb{C})$. There there exists a unique $w \in S_n$ and unique $c_1, \dots, c_\ell \in \mathbb{F}$ and unique $d_1, \dots, d_n \in \mathbb{F}^\times$ and unique $a_{ij} \in \mathbb{F}$ for $i, j \in R^+$ such that

$$g = y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) \cdot h_1(d_1) \cdots h_n(d_n) \cdot \left(\prod_{i < j} x_{ij}(a_{ij}) \right),$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is the chosen reduced word for w and the product $\left(\prod_{i < j} x_{ij}(a_{ij}) \right)$ is taken in the specific chosen order on $\{\text{pairs } (i, j) \text{ with } i, j \in \{1, \dots, n\} \text{ and } i < j\}$.

The next goal is to state the theorem in terms of the root system notations of Section 4.4. Let $w \in S_n$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w . The root sequence for the reduced word for w^{-1} given by $w^{-1} = s_{i_\ell} \cdots s_{i_1}$ is $(\beta_1, \dots, \beta_\ell)$ given by

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1} \alpha_{i_2}, \quad \dots \quad \beta_\ell = s_{i_1} s_{i_2} \cdots s_{i_{\ell-1}} \alpha_{i_\ell},$$

and, by Proposition 4.6, $\{\beta_1, \dots, \beta_\ell\} = \text{Inv}(w^{-1}) \subseteq R^+$. Then

$$\begin{aligned} y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) &= x_{\alpha_{i_1}}(c_1) s_{i_1} \cdots x_{i_\ell}(c_\ell) s_{i_\ell} \\ &= x_{\alpha_{i_1}}(c_1) x_{s_{i_1} \alpha_{i_2}}(c_2) \cdots x_{s_{i_1} \cdots s_{i_{\ell-1}} \alpha_{i_\ell}}(c_\ell) s_{i_1} \cdots s_{i_\ell} \\ &= x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) w, \end{aligned}$$

with $x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) \in B$. Thus, if $b \in GL_n(\mathbb{F})$ is upper triangular then $y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) b \in BwB$.

Theorem 5.3. *Suppose that for each $w \in S_n$ a fixed reduced word for w has been chosen. Also choose a specific order on R^+ , where R^+ is the set of positive roots defined in (Rpdef).*

Let $g \in GL_n(\mathbb{C})$. There there exists a unique $w \in S_n$ and unique $c_1, \dots, c_\ell \in \mathbb{F}$ and unique $d_1, \dots, d_n \in \mathbb{F}^\times$ and unique $a_\alpha \in \mathbb{F}$ for $\alpha \in R^+$ such that

$$g = x_{\beta_1}(c_1) \cdots x_{\beta_\ell}(c_\ell) \cdot w \cdot h_1(d_1) \cdots h_n(d_n) \cdot \left(\prod_{\alpha \in R^+} x_\alpha(a_\alpha) \right),$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is the chosen reduced word for w , the root sequence for the reduced word $w^{-1} = s_{i_\ell} \cdots s_{i_1}$ is $(\beta_1, \dots, \beta_\ell)$, and the product $\left(\prod_{\alpha \in R^+} x_\alpha(a_\alpha) \right)$ is in the specific chosen order on R^+ .

5.2 The Bruhat decomposition

The following structural result is a direct consequence of the normal form results of Section [5.1](#).

Theorem 5.4. (*Bruhat decomposition*) Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$. Let $G = GL_n(\mathbb{F})$. Then

$$G = \bigsqcup_{w \in W} BwB. \quad (\text{Bdecomp})$$

Let $w \in S_n$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w . Then

$$BwB = \bigsqcup_{c_1, \dots, c_\ell \in \mathbb{F}} y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell)B. \quad (\text{Bcell})$$

Example 5.1. When $n = 3$, coset representatives of cosets of B in the double cosets in $B \backslash G / B$

$$\begin{aligned} B1B &= \{B\}, \\ Bs_1B &= \{y_1(c)B \mid c \in \mathbb{F}_q\}, \\ Bs_2B &= \{y_2(c)B \mid c \in \mathbb{F}_q\}, \\ Bs_1s_2B &= \{y_1(c_1)y_2(c_2)B \mid c_1, c_2 \in \mathbb{F}_q\}, \\ Bs_2s_1B &= \{y_2(c_1)y_1(c_2)B \mid c_1, c_2 \in \mathbb{F}_q\}, \\ Bs_1s_2s_1B &= \{y_1(c_1)y_2(c_2)y_1(c_1)B \mid c_1, c_2, c_3 \in \mathbb{F}_q\}, \end{aligned}$$

and

$$\begin{aligned} B &= \{h_1(d_1)h_2(d_2)h_3(d_3)x_{23}(c_3)x_{13}(c_2)x_{12}(c_1) \mid c_1c_2c_3 \in \mathbb{F}_q^\times, d_1, d_2, d_3 \in \mathbb{F}_q^\times\} \\ &= \left\{ \begin{pmatrix} d_1 & d_1c_1 & d_1c_2 \\ 0 & d_2 & d_2c_3 \\ 0 & 0 & d_3 \end{pmatrix} \mid c_1c_2c_3 \in \mathbb{F}_q^\times, d_1, d_2, d_3 \in \mathbb{F}_q^\times \right\}, \end{aligned}$$

so that $\text{Card}(B) = (q-1)^3 q^3$. □

5.3 The Hecke algebra $H_n(q)$ and the flag representation $\mathbf{1}_B^G$

Let $G = GL_n(\mathbb{F}_q)$ and let B be the subgroup of upper triangular matrices in G . The *Hecke algebra* is the subalgebra $H_n(q)$ of $\mathbb{C}G$ given by

$$H_n(q) = \left\{ f = \sum_{g \in G} f(g)g \mid \text{if } b_1, b_2 \in B \text{ then } b_1fb_2 = f \right\}. \quad (\text{Hckdef})$$

Define

$$T_w = \frac{1}{|B|} \sum_{g \in BwB} g, \quad \text{for } w \in S_n.$$

By [\(Bdecomp\)](#),

$$\{T_w \mid w \in S_n\}, \quad \text{is a basis of } H_n(q).$$

The *flag representation* is the vector space

$$\mathbf{1}_B^G = \left\{ v = \sum_{g \in G} v(g)g \mid \text{if } b \in B \text{ then } vb = v \right\}. \quad (\text{1BGdef})$$

Define

$$v_y = \frac{1}{|B|} \sum_{g \in yB} g, \quad \text{for } y \in G.$$

By Bcell,

$$\left\{ v_{y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell)} \mid \begin{array}{l} w \in S_n \text{ and } c_1, \dots, c_\ell \in \mathbb{F} \text{ and} \\ w = s_{i_1} \cdots s_{i_\ell} \text{ is the greedy reduced word for } w \end{array} \right\} \text{ is a basis of } \mathbf{1}_B^G.$$

The group G acts on $\mathbf{1}_B^G$ by left multiplication and the Hecke algebra $H_n(q)$ acts on $\mathbf{1}_B^G$ by right multiplication. Via these actions $\mathbf{1}_B^G$ is a module for $\mathbb{C}G \otimes H_n(q)$.

5.4 Action of $\mathbb{C}G$ and of $H_n(q)$ on $\mathbf{1}_B^G$

Let's write the explicit action of $H_n(q)$ on $\mathbf{1}_B^G$. Recall that if $y \in G$ then

$$v_y = \frac{1}{|B|} \sum_{x \in yB} x, \quad \text{so that} \quad v_{yb} = v_y \text{ if } b \in B.$$

The action of $\mathbb{C}G$ on $\mathbf{1}_B^G$ is given by

$$gv_y = v_{gy}, \quad \text{for } g \in G. \quad (\text{Gon1BG})$$

The following proposition provides an explicit computation of the action of $H_n(q)$ on $\mathbf{1}_B^G$.

Proposition 5.5. *Let $w \in W$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w . Let $c_1, \dots, c_\ell \in \mathbb{F}_q$.*

(a) *If $\ell(ws_i) > \ell(w)$ then*

$$v_{y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell)} T_{s_i} = \sum_{c \in \mathbb{F}_q} v_{y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) y_i(c)}$$

(b) *If $\ell(ws_i) < \ell(w)$ then assume that $i_\ell = i$ and get*

$$v_{y_{i_1}(c_1) \cdots y_{i_{\ell-1}}(c_{\ell-1}) y_i(c_\ell)} T_{s_i} = v_{y_{i_1}(c_1) \cdots y_{i_{\ell-1}}(c_{\ell-1})} + \sum_{c \in \mathbb{F}_q^\times} v_{y_{i_1}(c_1) \cdots y_{i_{\ell-1}}(c_{\ell-1}) y_i(c_{\ell-1} c^{-1})}$$

Proof. If $b' \in B$ then $b' T_{s_i}$ and so

$$v_y T_{s_i} = \frac{1}{|B|} \sum_{b' \in B} y b' T_{s_i} = \frac{1}{|B|} \sum_{b' \in B} y T_{s_i} = y T_{s_i} = \frac{1}{|B|} \left(\sum_{c \in \mathbb{F}_q} \sum_{b \in B} y y_i(c) b \right) = \sum_{c \in \mathbb{F}_q} v_{y y_i(c)}.$$

(a) If $\ell(ws_i) > \ell(w)$ then

$$Bws_i B = \bigsqcup_{c_1, \dots, c_\ell, c \in \mathbb{F}_q} y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) y_i(c) B,$$

so that $\{y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) y_i(c) \mid c_1, \dots, c_\ell, c \in \mathbb{F}_q\}$ is a set of representatives of the B cosets in $Bws_i B$.

(b) If $\ell(ws_i) < \ell(w)$ then the reflection equation refrel gives that

$$\begin{aligned} v_{y_{i_1}(c_1) \cdots y_{i_{\ell-1}}(c_{\ell-1}) y_i(c_\ell)} T_i &= \sum_{c \in \mathbb{F}} v_{y_{i_1}(c_1) \cdots y_{i_{\ell-1}}(c_{\ell-1}) y_i(c_\ell) y_i(c)} \\ &= v_{y_{i_1}(c_1) \cdots y_{i_{\ell-1}}(c_{\ell-1})} + \sum_{c \in \mathbb{F}^\times} v_{y_{i_1}(c_1) \cdots y_{i_{\ell-1}}(c_{\ell-1}) y_i(c_{\ell-1} c^{-1})}. \end{aligned}$$

□

5.5 The product in the Hecke algebra

The following proposition provides an explicit computation of the product in $H_n(q)$.

Proposition 5.6. *For $w \in W$ and $i \in \{1, \dots, n\}$,*

$$T_w T_{s_i} = \begin{cases} (q-1)T_w + qT_{ws_i}, & \text{if } \ell(ws_i) = \ell(w) - 1, \\ T_{ws_i}, & \text{if } \ell(ws_i) = \ell(w) + 1. \end{cases}$$

Proof. We know

$$T_w = \sum_{c_1, \dots, c_\ell} v_{y_{i_1}(c_1) \dots y_{i_\ell}(c_\ell)}.$$

If $\ell(ws_i) = \ell(w) + 1$ then Proposition 5.5(a) gives

$$T_w T_{s_i} = \sum_{c_1, \dots, c_\ell} v_{y_{i_1}(c_1) \dots y_{i_\ell}(c_\ell)} T_{s_i} = \sum_{c_1, \dots, c_\ell, c} v_{y_{i_1}(c_1) \dots y_{i_\ell}(c_\ell) y_i(c)} = T_{ws_i}.$$

If $\ell(ws_i) = \ell(w) - 1$ then Proposition 5.5(b) gives

$$\begin{aligned} T_w T_{s_i} &= \sum_{c_1, \dots, c_\ell} v_{y_{i_1}(c_1) \dots y_{i_\ell}(c_\ell)} T_{s_i} \\ &= \sum_{c_1, \dots, c_\ell} \left(v_{y_{i_1}(c_1) \dots y_{i_{\ell-1}}(c_{\ell-1})} + \sum_{c \in \mathbb{F}^\times} v_{y_{i_1}(c_1) \dots y_{i_{\ell-1}}(c_{\ell-1}) y_i(c_{\ell-1}^{-1}c)} \right) \\ &= \sum_{c_1, \dots, c_{\ell-1}} \left(\sum_{c_\ell \in \mathbb{F}_q} v_{y_{i_1}(c_1) \dots y_{i_{\ell-1}}(c_{\ell-1})} \right) + \sum_{c_1, \dots, c_{\ell-1}} \left(\sum_{d \in \mathbb{F}_q} \sum_{\substack{c_\ell \in \mathbb{F}_q \\ c_\ell \neq d}} v_{y_{i_1}(c_1) \dots y_{i_{\ell-1}}(c_{\ell-1}) y_i(d)} \right) \\ &= q \sum_{c_1, \dots, c_{\ell-1}} v_{y_{i_1}(c_1) \dots y_{i_{\ell-1}}(c_{\ell-1})} + (q-1) \sum_{c_1, \dots, c_{\ell-1}, d \in \mathbb{F}_q} v_{y_{i_1}(c_1) \dots y_{i_{\ell-1}}(c_{\ell-1}) y_i(d)} \\ &= qT_{ws_i} + (q-1)T_w. \end{aligned}$$

□

5.6 The flag variety, Schubert varieties and the Bruhat order

Let $\mathbb{F}$ be a field. For $i \in \{1, \dots, n-1\}$ and $c \in \mathbb{F}$, define

$$y_i(c) = 1 + (c-1)E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} \quad \text{and define} \quad y_i(\infty) = 1.$$

The *flag variety* is

$$G/B = \{gB \mid g \in GL_n(\mathbb{F}_q)\}.$$

Let $w \in S_n$ and let $w = s_{i_1} \dots s_{i_\ell}$ be a reduced word for w . The *Schubert variety indexed by w in the flag variety* is

$$\overline{BwB} = \{y_{i_1}(c_1) \dots y_{i_\ell}(c_\ell) B \mid c_1, \dots, c_\ell \in \mathbb{F} \cup \{\infty\}\} \quad (\text{Svar})$$

By Theorem 4.5 and the relations in (bldrel), the Schubert variety $\overline{BwB}$ does not depend on the choice of the reduced word for w . The *Bruhat order* is the partial order $\leq$ on S_n defined by

$$\overline{BwB} = \bigsqcup_{v \leq w} BvB. \quad (\text{Bhtord})$$

5.6.1 Partial flag varieties, minimal length representatives and the Bruhat order on W^μ

Let $\mu = (\mu_1, \dots, \mu_\ell)$ with $\mu_1, \dots, \mu_\ell \in \mathbb{Z}_{>0}$ and $\mu_1 + \dots + \mu_\ell = n$. Let

$$S_\mu = S_{\mu_1} \times \dots \times S_{\mu_\ell} \quad \text{and let} \quad J(\mu) = \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \dots + \mu_\ell\}.$$

Then elements

$$W^\mu = \{w \in S_n \mid \text{if } i \notin J(\mu) \text{ then } w(i) < w(i+1)\}.$$

are a set of minimal length representatives of the cosets in S_n/S_μ .

$$\text{The } \mu\text{-Bruhat decomposition is} \quad G = \bigsqcup_{w \in W^\mu} BwP_\mu.$$

The μ -partial flag variety is

$$G/P_\mu = \{gP_\mu \mid g \in GL_n(\mathbb{F})\}.$$

Let $w \in W^\mu$ and let $w = s_{i_1} \dots s_{i_\ell}$ be a reduced word for w . The *Schubert variety for w in the μ -partial flag variety* is

$$\overline{BwP_\mu} = \{y_{i_1}(c_1) \dots y_{i_\ell}(c_\ell)P_\mu \mid c_1, \dots, c_\ell \in \mathbb{F} \cup \{\infty\}\}$$

The *Bruhat order on W^μ* is the partial order on W^μ defined by

$$\overline{BwP_\mu} = \bigsqcup_{v \leq w} BvP_\mu.$$

5.6.2 Grassmannians, partitions in a $k \times n$ rectangle and the dominance order

If $\mu = (k, n-k)$ then the μ -partial flag variety is the *Grassmannian of k dimensional subspaces of $\mathbb{F}^n$* ,

$$\mathbb{G}(\mathbb{F}^n)_k = G/P_{(k, n-k)} = \{gP_{(k, n-k)} \mid g \in GL_n(\mathbb{F})\}.$$

For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1, \dots, \lambda_k \in \mathbb{Z}_{\geq 0}$ and $n \geq \lambda_1 \geq \dots \geq \lambda_k$ define a permutation $w_\lambda \in S_n$ by

$$\begin{aligned} & \text{if } i \in \{1, \dots, n\} \text{ and } i \neq k \text{ then } w(i) < w(i+1), \text{ and} \\ w_\lambda(1) &= \lambda_k + 1, \quad w_\lambda(2) = \lambda_{k-1} + 2, \quad \dots, \quad w_\lambda(k-1) = \lambda_2 + k - 1, \quad w_\lambda(k) = \lambda_1 + k. \end{aligned}$$

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When $\mu = (k, n-k)$ then $S_{(k, n-k)} = S_k \times S_{n-k}$ and the map

$$\begin{array}{ccc} \{\text{partitions } \lambda \text{ with } \lambda \subseteq (n^k)\} & \leftrightarrow & W^{(n, n-k)} \\ \lambda & \mapsto & w_\lambda \end{array} \quad \text{is a bijection.}$$

Via this bijection, the Schubert varieties $\overline{Bw_\lambda P_{(k, n-k)}}$ in the Grassmannian $\mathbb{G}_k(\mathbb{F}^n)$ are indexed by partitions λ such that $\lambda \subseteq (n^k)$. The *dominance order on $\{\text{partitions } \lambda \text{ with } \lambda \subseteq (n^k)\}$* is given by setting $\mu \leq \lambda$ if μ and λ satisfy the conditions

$$\text{if } i \in \{1, \dots, k\} \text{ then } \mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i. \quad (\text{Gdomorder})$$

If λ is a partition and $\lambda \subseteq (n^k)$ then

$$\overline{Bw_\lambda P_{(k, n-k)}} = \bigsqcup_{\substack{\mu \subseteq (n^k) \\ \mu \leq \lambda}} Bw_\mu P_{(k, n-k)}.$$