22 Generators and relations for S_n and $GL_n(\mathbb{F})$

22.1 A presentation theorem for S_n

Let S_n be the symmetric group of permutation matrices and let

$$s_i = 1 + E_{i,i+1} + E_{i+1,i} - E_{ii} - E_{i+1,i+1}, \qquad \text{for } i \in \{1, \ldots, n-1\}.$$

The following theorem shows that the symmetric group S_n is a Coxeter group.

Theorem 22.1. *The symmetric group S_n is presented by generators $s_1, \ldots, s_{n-1}$ and relations*

$$s_j^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_j s_k = s_k s_j, \tag{Crels}$$

for $j, k \in \{1, \ldots, n-1\}$ with $k \notin \{j-1, j+1\}$ and $i \in \{1, \ldots, n-2\}$.

Proof sketch. The proof requires four steps:

(1) Generators A in terms of generators B.

(2) Generators B in terms of generators A.

(3) Relations A from relations B.

(4) Relations B from relations A.

Here

Generators A: { permutation matrices}

Relations A: { matrix multiplication of permutation matrices}

Generators B: { simple transpositions $s_1, \ldots, s_{n-1}$ }

Relations B: { the relations in (Crels) }

Further details of the proof are given in Section 22.1.3. □

22.1.1 Length and reduced words

Let $w \in S_n$. A reduced word for w is an expression $w = s_{i_1} \cdots s_{i_\ell}$ with $i_1, \ldots, i_\ell \in \{1, \ldots, n-1\}$ and ℓ minimal.

The *length* of w is $\ell(w)$, the length of a reduced word for w.

Proposition 22.2. *Let*

$$\mathrm{Inv}(w) = \{(i,j) \mid i, j \in \{1, \ldots, n\} \text{ with } i < j \text{ and } w(i) > w(j)\}.$$

Then

$$\ell(w) = \mathrm{Card}(\mathrm{Inv}(w)).$$

22.1.2 A reduced word algorithm for $w \in S_n$

Let $w \in S_n$. The following is an explicit algorithm for producing a reduced word for w.
Let $j_1 > 1$ be maximal such that $w_{j,1} \neq 0$. Let

$$w^{(1)} = \begin{cases} w, & \text{if } j_1 \text{ does not exist,} \\ s_1 \cdots s_{j_1-1} w, & \text{if } j_1 \text{ exists.} \end{cases}$$

Let $j_2 > 2$ be maximal such that $w^{(1)}_{j,2} \neq 0$. If j_2 does not exist set $w^{(2)} = w^{(1)}$ and if j_2 does exist set

$$w^{(2)} = \begin{cases} w^{(1)}, & \text{if } j_2 \text{ does not exist,} \\ s_2 \cdots s_{j_2-1} w^{(1)}, & \text{if } j_2 \text{ exists.} \end{cases}$$

Continue this process to produce $w^{(1)}, \ldots, w^{(n)}$. Then $w^{(n)} = 1$ and

$$w = \cdots (s_{j_2-1} \cdots s_2)(s_{j_1-1} \cdots s_1) \quad \text{is a reduced word for } w. \tag{gdyredwd}$$

22.1.3 Proof of the presentation theorem for S_n

The *simple transpositions in S_n* are the matrices $s_i = s_{i,i+1}$,

$$s_i = \begin{array}{c|c} & \begin{matrix} & & & i & i+1 & & & \end{matrix} \\ \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \end{matrix} & \begin{pmatrix} 1 &&&&&&& \\ & \ddots &&&&&& \\ && 1 &&&&& \\ &&& 0 & 1 &&& \\ &&& 1 & 0 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix} \end{array}, \qquad \text{for } i \in \{1, \ldots, n-1\}. \tag{22.1}$$

Proposition 22.3. *The symmetric group S_n is presented by generators $s_1, s_2, \ldots, s_{n-1}$ and relations*

$$s_i^2 = 1 \qquad \textit{and} \qquad s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1} \qquad \textit{and} \qquad s_k s_\ell = s_\ell s_k, \tag{22.2}$$

for $i, j, k, \ell \in \{1, \ldots, n-1\}$ with $j \neq n-1$ and $k \neq \ell \pm 1$.

Proof.
Generators A: the set of permutation matrices.
Relations A: all products of permutations $w_1 w_2$ given by matrix multiplication.
Generators B: $s_1, \ldots, s_{n-1}$.
Relations B: As given in (22.2).

The proof is accomplished in four steps:

(1) Write generators B in terms of generators A.
(2) Deduce relations B from relations A.
(3) Write generators A in terms of generators B.
(4) Deduce relations A from relations B.

Step 1: Generators B in terms of generators A. This is provided by (22.1).

Step 2: Relations B from relations A. This step is given the following matrix computations:

$$s_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$s_1 s_2 s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$s_2 s_1 s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so that $s_1 s_2 s_1 = s_2 s_1 s_2$ and

$$s_1 s_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$s_3 s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

so that $s_1 s_3 = s_3 s_1$.

Step 3: Generators A in terms of generators B.

Let $w \in S_n$.
Let $j_1 \in \{1, \ldots, n\}$ be such that $w(j_1, 1) = 1$ and let $w^{(1)} = s_1 s_2 \cdots s_{j_1-1} w$.
Let $j_2 \in \{2, \ldots, n\}$ be such that $w^{(1)}(j_2, 2) = 1$ and let $w^{(2)} = s_2 s_3 \cdots s_{j_2-1}$.
Continue this process to obtain

$$\cdots \ (s_2 s_3 \cdots s_{j_2-1})(s_1 s_2 \cdots s_{j_1-1}) w = 1.$$

Thus

$$w = (s_{j_1-1} \cdots s_2 s_1)(s_{j_2-1} \cdots s_3 s_2) \cdots .$$

The expression for w is a reduced word for w and a subword of the reduced word of the longest element given by

$$(s_{n-1} \cdots s_2 s_1)(s_{n-1} \cdots s_3 s_2) \cdots (s_{n-1} s_{n-2}) s_{n-1} = w_0.$$

Step 4: Relations A from relations B.

$$\begin{aligned} s_i(s_{j-1} \cdots s_2 s_1) &= s_{j-1} \cdots s_{i+2} s_i s_{i+1} s_i s_{i-1} \cdots s_2 s_1, \quad \text{by the third set of relations in (22.2)}, \\ &= s_{j-1} \cdots s_{i+2} s_{i+1} s_i s_{i+1} s_{i-1} \cdots s_2 s_1, \quad \text{by the second set of relations in (22.2)}, \\ &= (s_{j-1} \cdots s_{i+2} s_{i+1} s_i s_{i-1} \cdots s_2 s_1) s_i, \quad \text{by the third set of relations in (22.2)}, \end{aligned}$$

So $s_i w$ can be written in normal form. By Step 3, w_1 can be written as a product of simple transpositions, so one simple transposition at a time, $w_1 w$ can be written in normal form. □

If

$$w = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{then} \quad s_3(s_2s_3)(s_1s_2w) = s_3(s_2s_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = s_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 1,$$

so that $w = (s_2s_1)(s_3s_2)s_3$.

22.1.4 The graph of reduced words for $w \in S_n$

Define a graph $\Gamma(w)$ with

Vertices: $\{\text{reduced words of } w\}$
Edges: $u \to u'$ if $u' = s_{i_1} \cdots s_{i_\ell}$ is obtained from $u = s_{j_1} \cdots s_{j_\ell}$ by applying a relation $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ or a relation $s_is_j = s_js_i$ with $j \notin \{i-1, i+1\}$.

Theorem 22.4. *Let $w \in S_n$. The graph $\Gamma(w)$ of reduced words of w is connected.*

Proof. Let

$$w = s_{i_1} \cdots s_{i_\ell} \qquad \text{and} \qquad w = s_{j_1} \cdots s_{j_\ell}$$

be reduced words.

Case 1: $i_1 = j_1$. The two reduced words for w have the same first letter. By induction, the reduced words $v = s_{i_2} \cdots s_{i_\ell}$ and $v = s_{j_2} \cdots s_{j_\ell}$ are connected.

Case 2: $i_1 \neq j_1$. Since $\ell(s_{j_1 w}) < \ell(w)$ then there exists k such that $s_{j_1}w = s_{i_1} \cdots s_{i_{k-1}} \cancel{s_{i_k}} s_{i_{k+1}} \cdots s_{i_\ell}$.

Case 2a: $k \neq \ell$. Then

$$\begin{aligned} w &= s_{j_1} \cdots s_{j_\ell} \\ w &= s_{j_1}s_{i_1} \cdots s_{i_{k-1}} \cancel{s_{i_k}} s_{i_{k+1}} \cdots s_{i_\ell} \quad \text{and} \\ w &= s_{i_1} \cdots s_{i_\ell} \end{aligned}$$

are all reduced words for w. Since the first factor is the same in the first two of these they are connected. Since the last factor is the same in the last two of these they are connected. So, by transitivity, the first is connected to the last.

Case 2b: $k = \ell$ and $j_1 \notin \{i_1 - 1, i_1 + 1\}$. Then

$$\begin{aligned} w &= s_{j_1} \cdots s_{j_\ell}, \\ w &= s_{j_1}s_{i_1} \cdots s_{i_{\ell-1}}, \\ w &= s_{i_1}s_{j_1} \cdots s_{i_{\ell-1}} \quad \text{and} \\ w &= s_{i_1}s_{i_2} \cdots s_{i_\ell} \end{aligned}$$

and the first two are connected since they have the same first letter, the middle two are connected by the move $s_{j_1}s_{i_1} = s_{j_1}s_{i_1}$ and the last two are connected since they have the same first letter.

Case 2c: $k = \ell$ and $j_1 \in \{i_1 - 1, i_1 +\}$. Then

$$\begin{aligned} w &= s_{i_1}s_{i_2} \cdots s_{i_\ell}, \\ w &= s_{i_1}s_{j_1}s_{i_1} \cdots s_{i_{r-1}} \cancel{s_{i_r}} s_{i_{r+1}} \cdots s_{i_{\ell-1}}, \\ w &= s_{j_1}s_{i_1}s_{j_1} \cdots s_{i_{r-1}} \cancel{s_{i_r}} s_{i_{r+1}} \cdots s_{i_{\ell-1}}, \quad \text{and} \\ w &= s_{j_1}s_{j_2} \cdots s_{j_\ell}, \end{aligned}$$

and the first two are connected since they have the same first letter, the middle two are connected by the move $s_{i_1}s_{j_1}s_{i_1} = s_{j_1}s_{i_1}s_{j_1}$ and the last two are connected since they have the same first letter. $\square$

22.2 A presentation theorem for $GL_n(\mathbb{F})$

Let $\mathbb{F}$ be a field, let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{F})$ be the set of $n \times n$ matrices with entries in $\mathbb{F}$.

- An $n \times n$ *invertible matrix* is an $n \times n$ matrix $A \in M_n(\mathbb{F})$ such that

$$\text{there exists } A^{-1} \in M_n(\mathbb{F}) \quad \text{such that} \quad A^{-1}A = 1 \text{ and } AA^{-1} = 1.$$

- The *general linear group* is

$$GL_n(\mathbb{F}) = \{n \times n \text{ invertible matrices with entries in } \mathbb{F}\}.$$

The invertible elements of the field $\mathbb{F}$ are the elements of

$$\mathbb{F}^\times = \{d \in \mathbb{F} \mid d \neq 0\} = \{1 \times 1 \text{ invertible matrices with entries in } \mathbb{F}\} = GL_1(\mathbb{F}).$$

Theorem 22.5. *The group $GL_n(\mathbb{F})$ is presented by generators*

$$y_i(c), \quad h_j(d), \quad x_{k\ell}(c), \qquad \text{for} \qquad \begin{matrix} c \in \mathbb{F}, d_1, \ldots, d_n \in \mathbb{F}^\times, \\ i \in \{1, \ldots, n-1\}, j \in \{1, \ldots, n\} \\ k, \ell \in \{1, \ldots, n\} \text{ with } k < \ell. \end{matrix} \tag{GensB}$$

with the following relations:

- *The* reflection relation *is*

$$y_i(c_1)y_i(c_2) = \begin{cases} y_i(c_1 + c_2^{-1})h_i(c_2)h_{i+1}(-c_2^{-1})x_{i,i+1}(c_2^{-1}), & \text{if } c_2 \neq 0, \\ x_{i,i+1}(c_1), & \text{if } c_2 = 0. \end{cases} \tag{refrel}$$

- *The* building relation *is*

$$y_i(c_1)y_{i+1}(c_2)y_i(c_3) = y_{i+1}(c_3)y_i(c_1c_3 + c_2)y_{i+1}(c_1). \tag{bldrel}$$

- *The* x-interchange relations *are*

$$\begin{aligned} &x_{ij}(c_1)x_{ij}(c_2) = x_{ij}(c_1 + c_2), \\ &x_{ij}(c_1)x_{ik}(c_2) = x_{ik}(c_2)x_{ij}(c_1), && x_{ik}(c_1)x_{jk}(c_2) = x_{jk}(c_2)x_{ik}(c_1), \\ &x_{ij}(c_1)x_{jk}(c_2) = x_{jk}(c_2)x_{ij}(c_1)x_{ik}(c_1c_2), && x_{jk}(c_1)x_{ij}(c_2) = x_{ij}(c_2)x_{jk}(c_1)x_{ik}(-c_1c_2), \end{aligned} \tag{xint}$$

  where $i < j < k$.

- *The* h-processing relations *are*

$$h_i(d_1)h_j(d_2) = h_j(d_2)h_i(d_1) \qquad \text{and} \qquad h_i(d_1)h_i(d_2) = h_i(d_1d_2), \tag{hhrel}$$

- *Letting $h(d_1, \ldots, d_n) = h_1(d_1) \cdots h_n(d_n)$, the* h-past-y relation *is*

$$h(d_1, \ldots d_n)y_i(c) = y_i(cd_id_{i+1}^{-1})h(d_1, \ldots, d_{i-1}, d_{i+1}, d_i, d_{i+2}, \ldots, d_n). \tag{hpyrel}$$

- *Letting $h(d_1, \ldots, d_n) = h_1(d_1) \cdots h_n(d_n)$, the* h-past-x relation *is*

$$h(d_1, \ldots, d_n)x_{ij}(c) = x_{ij}(cd_id_j^{-1})h(d_1, \ldots, d_n). \tag{hpxrel}$$

- *The* x-past-y relations *are*

$$\begin{aligned} x_{i,i+1}(c_1)y_i(c_2) &= y_i(c_1+c_2)x_{i,i+1}(0), \\ x_{ik}(c_1)y_k(c_2) = y_k(c_2)x_{ik}(c_1c_2)x_{i,k+1}(c_1), &\qquad x_{i,k+1}(c_1)y_k(c_2) = y_k(c_2)x_{ik}(c_1), \\ x_{ij}(c_1)y_i(c_2) = y_i(c_2)x_{i+1,j}(c_1), &\qquad x_{i+1,j}(c_1)y_i(c_2) = y_i(c_2)x_{ij}(c_1)x_{i+1,j}(-c_1c_2), \end{aligned} \qquad \text{(xpyrel)}$$

where $i < k$ *and* $i+1 < j$.

Proof sketch.

Generators A: { invertible matrices}

Relations A: { matrix multiplication of invertible matrices}

Generators B: { row reducers, diagonal generators and elementary matrices }

Relations B: { the interchange relations in the statement }

The proof requires four steps:

(1) Generators A in terms of generators B.
(2) Generators B in terms of generators A.
(3) Relations A from relations B.
(4) Relations B from relations A.

Step(2), which requires the expression of the Generators B in terms of the generators A, is provided by the definitions in section 22.2.1.

Step (4), which derives the Relations B from relations A (matrix multiplicaton), is checked in section 22.2.3.

Step (1), which describes how to write an invertible matrix in terms of the elementary matrices is given in section 22.2.2.

Step (3), which descirbes how to derive Relations A (matrix multiplication) from the relations B, is checked in section 22.2.4.

□

22.2.1 Elementary matrices and row reducers

Let

$$E_{ij} \text{ be the } n\times n \text{ matrix with 1 in the } (i,j) \text{ entry and 0 elsewhere.}$$

- The *elementary matrices in* $GL_n(\mathbb{F})$ are the matrices

$$s_{ij} = 1 - E_{ii} - E_{jj} + E_{ij} + E_{ji}, \qquad \text{for } i,j\in\{1,\ldots,n\} \text{ with } i\ne j,$$

$$x_{ij}(c) = 1 + cE_{ij}, \qquad \text{for } i,j\in\{1,\ldots,n\} \text{ with } i\ne j \text{ and } c\in\mathbb{F},$$

$$h_i(d) = 1 + (d-1)E_{ii}, \qquad \text{for } i\in\{1,\ldots,n\} \text{ and } d\in GL_1(\mathbb{F}).$$

- The *row reducers* are $y_i(c) = x_{i,i+1}(c)s_{i,i+1}$ for $i \in \{1, \ldots, n-1\}$ and $c \in \mathbb{F}$.

$$y_i(c) = \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& c & 1 &&& \\ &&& 1 & 0 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix} \qquad \text{and} \qquad y_i(c)^{-1} = \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 &&&& \\ &&& 0 & 1 &&& \\ &&& 1 & -c &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix}$$

22.2.2 A reduced word algorithm for $g \in GL_n(\mathbb{F})$

Let $g \in GL_n(\mathbb{F})$ so that g is an $n \times n$ invertible matrix. The following is an explicit algorithm for writing g as a product of row reducers $y_i(c)$, diagonal generators $h_i(d)$ and upper triangular elementary matrices $x_{ij}(a_{ij})$. This procedure is no different than the usual row reduction procedure: namely, a way of writing an invertible matrix g in a 'normal form' as a product of elementary matrices by the 'row reduction' algorithm.

Let $j_1 > 1$ be maximal such that such that $g(j_1, 1) \neq 0$. Let If $j_1 = 1$ then let $g^{(1)} = g$. If $j_1 \neq 1$ then let

$$g^{(1)} = y_1\left(\tfrac{g(1,1)}{g(j_1,1)}\right)^{-1} y_2\left(\tfrac{g(1,2)}{g(j_1,1)}\right)^{-1} \cdots y_{j_1-1}\left(\tfrac{g(j_1-1,1)}{g(j_1,1)}\right)^{-1} g.$$

Let $j_2 > 2$ be maximal such that $g^{(1)}(j_2, 2) \neq 0$. If $j_2 = 2$ then let $g^{(2)} = g^{(1)}$. If $j_2 \neq 2$ then let

$$g^{(2)} = y_2\left(\tfrac{g^{(1)}(2,2)}{g^{(1)}(j_2,2)}\right)^{-1} y_3\left(\tfrac{g^{(1)}(3,2)}{g^{(1)}(j_2,2)}\right)^{-1} \cdots y_{j_2-1}\left(\tfrac{g^{(1)}(j_2-1,2)}{g^{(1)}(j_2,2)}\right)^{-1} g^{(1)}.$$

Continuing this process will produce $g^{(n)}$ which has the property that

the first nonzero entry in row $j+1$ is to the right of the first nonzero entry in row j.

Since g is invertible then $g^{(n)}$ must be upper triangular.

Let $b = g^{(n)}$. Then

$$\begin{aligned} g = \cdots (y_{j_2-1}\left(\tfrac{g^{(1)}(j_2-1,2)}{g^{(1)}(j_2,2)}\right) \cdots y_3\left(\tfrac{g^{(1)}(3,2)}{g^{(1)}(j_1,2)}\right) y_2\left(\tfrac{g^{(1)}(2,2)}{g^{(1)}(j_2,2)}\right)) \\ \cdot (y_{j_1-1}\left(\tfrac{g(j_1-1,1)}{g(j_1,1)}\right) \cdots y_2\left(\tfrac{g(2,1)}{g(j_1,1)}\right) y_1\left(\tfrac{g(1,1)}{g(j_1,1)}\right)) \cdot b \end{aligned}$$

The examples

$$x_{34}(c_{34})x_{24}(c_{24})x_{14}(c_{14})x_{23}(c_{23})x_{13}(c_{13})x_{12}(c_{12}) = \begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & 1 & c_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} &h_1(d_1)h_2(d_2)h_3(c_3)h_4(c_4)x_{34}(c_{34})x_{24}(c_{24})x_{14}(c_{14})x_{23}(c_{23})x_{13}(c_{13})x_{12}(c_{12}) \\ &= h_1(d_1)h_2(d_2)h_3(c_3)h_4(c_4) \begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & 1 & c_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d_1 & c_{12} & c_{13} & c_{14} \\ 0 & d_2 & c_{23} & c_{24} \\ 0 & 0 & d_3 & c_{34} \\ 0 & 0 & 0 & d_4 \end{pmatrix}. \end{aligned}$$

show how an upper triangular matrix is written in normal form as a product of $h_i(d)$ and $x_{jk}(c)$.

Example 22.1. *Let*

$$g = \begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Since

$$\begin{aligned}
g = \begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} &= y_2(\tfrac{1}{8}) \begin{pmatrix} 7 & 6 & 2 & 4 \\ 8 & 6 & 3 & 5 \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix} \\
&= y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & \frac{1}{4} & -\frac{3}{8} \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix} \\
&= y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) y_3(\tfrac{29}{4}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & \frac{1}{4} & -\frac{3}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix} \\
&= y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) y_3(\tfrac{29}{4}) y_2(\tfrac{3}{4}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{1}{2} & -\frac{15}{8} \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix} \\
&= y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) y_3(\tfrac{29}{4}) y_2(\tfrac{3}{4}) y_3(\tfrac{4}{5}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \\ 0 & 0 & 0 & -\frac{71}{40} \end{pmatrix}
\end{aligned}$$

then

$$g = y_2(\tfrac{1}{8}) y_1(\tfrac{7}{8}) y_3(\tfrac{29}{4}) y_2(\tfrac{3}{4}) y_3(\tfrac{4}{5}) h_1(8) h_2(1) h_3(-\tfrac{5}{8}) h_4(-\tfrac{71}{40}) x_{34}(-\tfrac{49}{8}) x_{24}(2) x_{14}(5) x_{23}(1) x_{13}(3) x_{12}(6)$$

is an expression for g purely in terms of the row reducers, the diagonal geberators and the upper triangular elementary matrices. $\square$

For $i, j \in \{1, \ldots, n\}$ with $i < j$ let $a_{ij} \in \mathbb{F}$. The product

$$\Big(\prod_{i<j} x_{ij}(a_{ij})\Big) \qquad \text{is in } \textit{matrix parametrization order} \text{ if}$$

$$x_{jk}(a_{jk}) \text{ appears before } x_{ik}(a_{ik}) \text{ for } j > i, \quad \text{and} \quad x_{j\ell}(a_{j\ell}) \text{ appears before } x_{ik}(a_{ik}) \text{ for } \ell > k.$$

Theorem 22.6. *Let $g \in GL_n(\mathbb{C})$. There there exists a unique $w \in S_n$ and unique $c_1, \ldots, c_\ell \in \mathbb{F}$ and unique $d_1, \ldots, d_n \in \mathbb{F}^\times$ and unique $a_{ij} \in \mathbb{F}$ for $i, j \in R^+$ such that*

$$g = y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell) \cdot h_1(d_1) \cdots h_n(d_n) \cdot \Big(\prod_{i<j} x_{ij}(a_{ij})\Big),$$

where $w = s_{i_1} \cdots s_{i_\ell}$ be the greedy reduced word for w and the product $\Big(\prod_{i<j} x_{ij}(a_{ij})\Big)$ is in matrix parametrization order.

22.2.3 Obtaining the interchange relations from matrix multiplication

Proof of the reflection relation:

If $c_1 \neq 0$ and $c_2 \neq 0$ then

$$\begin{aligned} y_1(c_1)y_1(c_2) &= \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1c_2+1 & c_1 \\ c_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1+c_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 0 & -c_2^{-1} \end{pmatrix} = \begin{pmatrix} c_1+c_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 0 \\ 0 & -c_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & c_2^{-1} \\ 0 & 1 \end{pmatrix} \\ &= y_1(c_1+c_2^{-1})h_1(c_2)h_2(-c_2^{-1})x_{12}(c_2^{-1}). \end{aligned}$$

If $c_2 = 0$ then

$$y_1(c_1)y_1(0) = \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} = x_{12}(c_1).$$

Proof of the building relation:

$$\begin{aligned} \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} c_1c_3+c_2 & 1 & 0 \\ c_3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1c_3+c_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

The computation for the proof of the first x-interchange relation is:

$$\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_1+c_2 \\ 0 & 1 \end{pmatrix}$$

The key computation for the proof of the h-past-y relation is:

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} cd_1 & d_1 \\ d_2 & 0 \end{pmatrix} = \begin{pmatrix} cd_1d_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_2 & 0 \\ 0 & d_1 \end{pmatrix}$$

Key computations for the proof of the x-past-y relations are:

$$\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1+c_2 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & c_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & c_1c_2 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c_1c_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & c_1 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & c_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & c_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c_1c_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The remaining relations are derived similarly.

22.2.4 Deriving matrix multiplication from the interchange relations

Suppose that g_1 and g_2 are two expressions given in the normal form of Theorem 22.6. The goal is to use the Relations B to rearrange and simplify the product g_1g_2.

Step 1. The h-past-x relations and h-past-y relations and the x-past-y relations allow us to move all the row reducers $y_i(c)$ to the left, all the elementary matrices $x_{ij}(c)$ to the right so that all the diagonal generators $h_i(d)$ are in the middle.

Step 2. The hh-relations allow us to write the product of the diagonal generators in the form $h(d_1, \ldots, d_n)$.

Step 3. The reflection relation and the building relation allow us to reduce the y-product to being a y-product for a reduced word of a permutation. Then the theorem that the graph of reduced words is conneted allows us to arrange this reduced word to be the greedy reduced word for w.

Step 4. The x-interchange relations allow us to put the $x_{ij}(c)$ into its appropriate place in the matrix presentation order.

In combination these moves rearrange the product g_1g_2 into normal form.

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