22 RSK correspondences and Schur functions

22.1 The Cauchy identity

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of elements of $\mathbb{Z}_{\geq 0}$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and there exists $\ell \in \mathbb{Z}_{>0}$ with $\lambda_\ell > 0$ and $\lambda_{\ell+1} = 0$. The *length of* λ is $\ell(\lambda) = \ell$.

Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{n\times m}(\mathbb{Z}_{\geq 0})$ be the set of $n \times m$ matrices with entries from $\mathbb{Z}_{\geq 0}$. For a partition λ define

$$B_n(\lambda) = \{\text{SSYTs } P \text{ of shape } \lambda \text{ filled from } \{1, \ldots, n\}\}.$$

If $\ell(\lambda) > n$ then $B_n(\lambda) = \emptyset$. The *RSK correspondence* is

$$\text{the bijection} \quad \begin{matrix} M_{n\times m}(\mathbb{Z}_{\geq 0}) & \longleftrightarrow & \bigsqcup_\lambda B_n(\lambda) \otimes B_m(\lambda) \\ A & \mapsto & (P, Q) \end{matrix} \quad \text{described in Section 22.2.} \qquad \text{(RSKbij)}$$

The *weight* of a matrix $A = (a_{ij})$ in $M_{n\times m}(\mathbb{Z}_{\geq 0})$ is

$$\mathrm{wt}(A) = \prod_{i=1}^n \prod_{j=1}^m (x_i y_j)^{a_{ij}}.$$

The *Schur function indexed by* λ *in the variables* $x_1, \ldots, x_n$ is

$$s_\lambda(x) = s_\lambda(x_1, \ldots, x_n) = \sum_{P \in B_n(\lambda)} x^{\mathrm{wt}(P)} = \sum_{P \in B_n(\lambda)} x_1^{\#\text{ 1s in } P} \cdots x_n^{\#\text{ ns in } P}.$$

If $\ell(\lambda) > n$ then $s_\lambda(x_1, \ldots, x_n) = 0$. The *Schur function indexed by* λ *in the variables* $y_1, \ldots, y_m$ is

$$s_\lambda(y) = s_\lambda(y_1, \ldots, x_m) = \sum_{Q \in B_m(\lambda)} y^{\mathrm{wt}(Q)} = \sum_{Q \in B_m(\lambda)} y_1^{\#\text{ 1s in } Q} \cdots y_m^{\#\text{ ms in } Q}.$$

Corollary 22.1. *(Cauchy identity)*

$$\prod_{j=1}^n \prod_{i=1}^m \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(x_1, \ldots, x_n) s_\lambda(y_1, \ldots, y_m).$$

Proof.

$$\begin{aligned}
\prod_{i=1}^n \prod_{j=1}^m \frac{1}{1 - x_i y_j} &= \prod_{i=1}^n \prod_{j=1}^m (1 + x_i y_j + (x_i y_j)^2 + (x_i y_j)^3 + \cdots) = \prod_{i=1}^n \prod_{j=1}^m \sum_{a_{ij} \in \mathbb{Z}_{\geq 0}} (x_i y_j)^{a_{ij}} \\
&= \sum_{A \in M_{m\times n}(\mathbb{Z}_{\geq 0})} \Big(\prod_{i=1}^n \prod_{j=1}^m (x_i y_j)^{a_{ij}} \Big) = \sum_{A \in M_{m\times n}(\mathbb{Z}_{\geq 0})} \mathrm{wt}(A) = \sum_\lambda x^{\mathrm{wt}(P)} y^{\mathrm{wt}(Q)} \\
&= \sum_\lambda s_\lambda(x_1, \ldots, x_n) s_\lambda(y_1, \ldots, y_m),
\end{aligned}$$

where the next to last equality is by the RSK correspondence. □

Restriction to matrices with column sums $(1, 1, \ldots, 1)$. Suppose that λ is a partition with m boxes. Then

$$\hat{S}_m^\lambda = \{\text{standard tableaux } Q \text{ of shape } \lambda\} = \{Q \in B_m(\lambda) \mid \mathrm{wt}(Q) = (1, 1, \ldots, 1)\}.$$

Restricting the RSK correspondence to the set

$$M_{m\times n}(\mathbb{Z}_{\geq 0})_{cs=(1,\ldots,1)} = \{A \in M_{m\times n}(\mathbb{Z}_{\geq 0}) \mid \mathrm{cs}(A) = (1,1,\ldots,1)\}$$

gives a bijection

$$\begin{matrix} M_{n\times m}(\mathbb{Z}_{\geq 0})_{\mathrm{cs}=(1,\ldots,1)} & \longleftrightarrow & \bigsqcup_\lambda B_n(\lambda)\otimes \hat{S}^\lambda_m \\ A & \mapsto & (P,Q) \end{matrix} \tag{CRSK}$$

and this bijection implies the identity

$$(x_1+\cdots+x_n)^m = \sum_{\lambda\in\mathbb{Y}_m} s_\lambda(x_1,\ldots,x_n)f_\lambda, \tag{CCI}$$

where f_λ is the number of standard tableaux of shape λ.

Restriction to permutation matrices. Suppose that $m = n$. Restricting the RSK correspondence to the set of matrices with row sums $(1,1,\ldots,1)$ and with column sums $(1,1,\ldots,1)$,

$$\begin{aligned} M_{n\times n}(\mathbb{Z}_{\geq 0})_{rs=cs=(1,\ldots,1)} &= \{A \in M_{n\times n}(\mathbb{Z}_{\geq 0}) \mid \mathrm{cs}(A) = \mathrm{rs}(A) = (1,1,\ldots,1)\} \\ &= \{n\times n \text{ permutation matrices}\} = S_n, \end{aligned}$$

gives a bijection

$$\begin{matrix} S_n & \longleftrightarrow & \bigsqcup_{\lambda\in\mathbb{Y}_n} \hat{S}^\lambda_n \otimes \hat{S}^\lambda_n \\ A & \mapsto & (P,Q) \end{matrix} \tag{RCRSK}$$

and this bijection implies the identity

$$n! = \sum_{\lambda\in\mathbb{Y}_n} f_\lambda^2, \tag{RCCI}$$

where f_λ is the number of standard tableaux of shape λ.

22.2 The RSK procedure

Identify a SSYT with a tensor product of columns so that, for example,

$$\begin{ytableau} 1 & 2 & 2 & 4 & 6 & 6 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 6 \\ 6 \end{ytableau} = \begin{ytableau} 1 \\ 3 \\ 4 \\ 6 \end{ytableau} \otimes \begin{ytableau} 2 \\ 3 \\ 6 \end{ytableau} \otimes \begin{ytableau} 2 \\ 4 \\ 6 \end{ytableau} \otimes \begin{ytableau} 4 \\ 5 \end{ytableau} \otimes \boxed{6} \otimes \boxed{6}.$$

Similarly, a pair of SSYT is a tensor product of pairs of column tableaux,

$$\left(\begin{ytableau} 1 & 1 & 1 & 2 & 9 \\ 2 & 3 & 7 & 7 \\ 3 & 8 & 8 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 2 & 4 & 6 & 6 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & 6 \\ 6 \end{ytableau} \right) = \left(\begin{ytableau} 1 \\ 2 \\ 3 \\ 7 \end{ytableau} \otimes \begin{ytableau} 1 \\ 3 \\ 8 \end{ytableau} \otimes \begin{ytableau} 1 \\ 7 \\ 8 \end{ytableau} \otimes \begin{ytableau} 2 \\ 7 \end{ytableau} \otimes \boxed{9} \otimes \boxed{6}\ ,\ \begin{ytableau} 1 \\ 3 \\ 4 \\ 6 \end{ytableau} \otimes \begin{ytableau} 2 \\ 3 \\ 6 \end{ytableau} \otimes \begin{ytableau} 2 \\ 4 \\ 6 \end{ytableau} \otimes \begin{ytableau} 4 \\ 5 \end{ytableau} \otimes \boxed{6} \otimes \boxed{6} \right)$$

$$= \left(\begin{ytableau} 1 \\ 2 \\ 3 \\ 7 \end{ytableau}, \begin{ytableau} 1 \\ 3 \\ 4 \\ 6 \end{ytableau} \right) \otimes \left(\begin{ytableau} 1 \\ 3 \\ 8 \end{ytableau}, \begin{ytableau} 2 \\ 3 \\ 6 \end{ytableau} \right) \otimes \left(\begin{ytableau} 1 \\ 7 \\ 8 \end{ytableau}, \begin{ytableau} 2 \\ 4 \\ 6 \end{ytableau} \right) \otimes \left(\begin{ytableau} 2 \\ 7 \end{ytableau}, \begin{ytableau} 4 \\ 5 \end{ytableau} \right) \otimes \left(\boxed{9}, \boxed{6} \right) \otimes \left(\boxed{6}, \boxed{6} \right) \tag{RSKa}$$

Finally, it is often convenient to identify a tensor product of pairs of single box tableaux as a two-line array:

$$(\boxed{3},\boxed{6}) \otimes (\boxed{3},\boxed{6}) \otimes (\boxed{7},\boxed{6}) \otimes (\boxed{8},\boxed{6}) \otimes (\boxed{3},\boxed{5}) = \begin{pmatrix} \boxed{6} & \boxed{6} & \boxed{6} & \boxed{6} & \boxed{5} \\ \boxed{3} & \boxed{3} & \boxed{7} & \boxed{8} & \boxed{3} \end{pmatrix}.$$

Identify a matrix $A = (a_{ij})$ as a two-line array by reading the entries of A in chinese reading order (vertically down columns, with columns taken right to left) with the pair $\begin{pmatrix} \boxed{j} \\ \boxed{i} \end{pmatrix}$ repeated a_{ij} times.

$$\begin{pmatrix} 0&2&0&1&0&0 \\ 1&0&1&0&0&0 \\ 0&0&0&0&1&2 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&2&0&1 \\ 0&0&0&0&0&2 \\ 0&0&1&0&0&0 \end{pmatrix} = \begin{pmatrix} \boxed{6} & \boxed{6} & \boxed{6} & \boxed{6} & \boxed{6} & \boxed{5} & \boxed{4} & \boxed{4} & \boxed{4} & \boxed{3} & \boxed{3} & \boxed{2} & \boxed{2} & \boxed{1} \\ \boxed{3} & \boxed{3} & \boxed{7} & \boxed{8} & \boxed{8} & \boxed{3} & \boxed{1} & \boxed{7} & \boxed{7} & \boxed{2} & \boxed{9} & \boxed{1} & \boxed{1} & \boxed{2} \end{pmatrix}$$

22.2.1 Generators of the RSK correspondence

Assume $0 = p_0 < p_1 < \cdots < p_k \le n$ and $0 = q_0 < q_1 < \cdots < q_k \le m$. Let $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. The RSK correspondence is generated by the following identifications:

$$\left(\boxed{i}, \boxed{j} \right) \otimes \left(\boxed{\begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{matrix}}, \boxed{\begin{matrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{matrix}} \right) = \left(\boxed{\begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_{\ell-1} \\ i \\ p_{\ell+1} \\ \vdots \\ p_k \end{matrix}}, \boxed{\begin{matrix} q_1 \\ q_2 \\ \vdots \\ q_{\ell-1} \\ q_\ell \\ q_{\ell+1} \\ \vdots \\ q_k \end{matrix}} \right) \otimes \left(\boxed{p_\ell}, \boxed{j} \right) \quad \text{if } p_{\ell-1} < i \le p_\ell,$$

$$\text{and} \quad \left(\boxed{i}, \boxed{j} \right) \otimes \left(\boxed{\begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{matrix}}, \boxed{\begin{matrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{matrix}} \right) = \left(\boxed{\begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_k \\ i \end{matrix}} \otimes \boxed{\begin{matrix} q_1 \\ q_2 \\ \vdots \\ q_k \\ j \end{matrix}} \right) \quad \text{if } p_\ell < i. \qquad \text{(RSKgen)}$$

To give an example, applying the steps in (RSKgen) to move the pair $(\boxed{3},\boxed{6})$ past the pair of SSYTs given in (RSKa) gives

$$
\begin{aligned}
&(\boxed{3},\boxed{6})\otimes\left(\begin{array}{ccccc}1&1&1&2&9\\2&3&7&7\\3&8&8\\7\end{array},\ \begin{array}{ccccc}1&2&2&4&6\\3&3&4&5\\4&6&6\\6\end{array}\right)
=(\boxed{3},\boxed{6})\otimes\left(\begin{array}{c}1\\2\\3\\7\end{array},\begin{array}{c}1\\3\\4\\6\end{array}\right)\otimes\left(\begin{array}{cccc}1&1&2&9\\3&7&7\\8&8\end{array},\ \begin{array}{cccc}2&2&4&6\\3&4&5\\6&6\end{array}\right)\\
&=\left(\begin{array}{c}1\\2\\{\color{red}3}\\7\end{array},\begin{array}{c}1\\3\\4\\6\end{array}\right)\otimes(\boxed{3},\boxed{6})\otimes\left(\begin{array}{cccc}1&1&2&9\\3&7&7\\8&8\end{array},\ \begin{array}{cccc}2&2&4&6\\3&4&5\\6&6\end{array}\right)
=\left(\begin{array}{cc}1&1\\2&{\color{red}3}\\{\color{red}3}&8\\7\end{array},\begin{array}{cc}1&2\\3&3\\4&6\\6\end{array}\right)\otimes(\boxed{3},\boxed{6})\otimes\left(\begin{array}{ccc}1&2&9\\7&7\\8\end{array},\ \begin{array}{ccc}2&4&6\\4&5\\6\end{array}\right)\\
&=\left(\begin{array}{ccc}1&1&1\\2&{\color{red}3}&{\color{red}3}\\{\color{red}3}&8&8\\7\end{array},\begin{array}{ccc}1&2&2\\3&3&4\\4&6&6\\6\end{array}\right)\otimes(\boxed{7},\boxed{6})\otimes\left(\begin{array}{cc}2&9\\7\end{array},\ \begin{array}{cc}4&6\\5\end{array}\right)
=\left(\begin{array}{cccc}1&1&1&2\\2&{\color{red}3}&{\color{red}3}&{\color{red}7}\\{\color{red}3}&8&8\\7\end{array},\begin{array}{cccc}1&2&2&4\\3&3&4&5\\4&6&6\\6\end{array}\right)\otimes(\boxed{7},\boxed{6})\otimes(\boxed{9},\boxed{6})\\
&=\left(\begin{array}{ccccc}1&1&1&2&{\color{red}7}\\2&{\color{red}3}&{\color{red}3}&{\color{red}7}\\{\color{red}3}&8&8\\7\end{array},\ \begin{array}{ccccc}1&2&2&4&6\\3&3&4&5\\4&6&6\\6\end{array}\right)\otimes(\boxed{9},\boxed{6})
=\left(\begin{array}{cccccc}1&1&1&2&{\color{red}7}&{\color{red}9}\\2&{\color{red}3}&{\color{red}3}&{\color{red}7}\\{\color{red}3}&8&8\\7\end{array},\ \begin{array}{cccccc}1&2&2&4&6&6\\3&3&4&5\\4&6&6\\6\end{array}\right).
\end{aligned}
\tag{RSKSTEP}
$$

At each step the 'inserted entry' is marked in red. The equality (RSKSTEP) is the first step in the example of the RSK correspondence given in Example 22.1. This example in Example 22.1. is displayed in 'reverse', from the pair of SSYTs to the nonnegative integer matrix, to illustrate that the steps can be reversed, which is what gives that the RSK correspondence is a bijection. At each equality the 'bumping sequence' is indicated in red in analogy with the example in (RSKSTEP). The result of Example 22.1 is that

$$
\begin{aligned}
&\left(\begin{array}{cccccc}1&1&1&2&7&9\\2&3&3&7\\3&8&8\\7\end{array},\ \begin{array}{cccccc}1&1&1&2&4&6\\3&3&4&5\\4&6&6\\6\end{array}\right)\\
&=\left(\begin{array}{cccccccccccccc}\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{5}&\boxed{4}&\boxed{4}&\boxed{4}&\boxed{3}&\boxed{3}&\boxed{2}&\boxed{2}&\boxed{1}\\ \boxed{3}&\boxed{3}&\boxed{7}&\boxed{8}&\boxed{8}&\boxed{3}&\boxed{1}&\boxed{7}&\boxed{7}&\boxed{2}&\boxed{9}&\boxed{1}&\boxed{1}&\boxed{2}\end{array}\right)\\
&=\begin{pmatrix}0&2&0&1&0&0\\1&0&1&0&0&0\\0&0&0&0&1&2\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&2&0&1\\0&0&0&0&0&2\\0&0&1&0&0&0\end{pmatrix}\qquad\text{a matrix in } M_{9\times 6}(\mathbb{Z}_{\ge 0})
\end{aligned}
$$

with weight

$$x_1^3x_2^2x_3^3x_7^3x_8^2x_9\cdot y_1^3y_2y_3^2y_4^3y_5y_6^4=(x_1y_2)^2(x_1y_4)(x_2y_1)x_2y_3)(x_3y_5)(x_3y_6)^2(x_7y_4)^2(x_7y_6)(x_8y_6)^2(x_9y_3).$$

Example 22.1.

$$\left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}&\boxed{7}&\boxed{9}\\ \boxed{2}&\boxed{3}&\boxed{3}&\boxed{7}\\ \boxed{3}&\boxed{8}&\boxed{8}\\ \boxed{7}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}&\boxed{4}&\boxed{6}&\boxed{6}\\ \boxed{3}&\boxed{3}&\boxed{4}&\boxed{5}\\ \boxed{4}&\boxed{6}&\boxed{6}\\ \boxed{6}\end{matrix}\right) = (\boxed{3},\boxed{6}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}&\boxed{9}\\ \boxed{2}&\boxed{3}&\boxed{7}&\boxed{7}\\ \boxed{3}&\boxed{8}&\boxed{8}\\ \boxed{7}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}&\boxed{4}&\boxed{6}\\ \boxed{3}&\boxed{3}&\boxed{4}&\boxed{5}\\ \boxed{4}&\boxed{6}&\boxed{6}\\ \boxed{6}\end{matrix}\right)$$

$$= (\boxed{3},\boxed{6}) \otimes (\boxed{3},\boxed{6}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}\\ \boxed{2}&\boxed{7}&\boxed{7}&\boxed{9}\\ \boxed{3}&\boxed{8}&\boxed{8}\\ \boxed{7}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}&\boxed{4}\\ \boxed{3}&\boxed{3}&\boxed{4}&\boxed{5}\\ \boxed{4}&\boxed{6}&\boxed{6}\\ \boxed{6}\end{matrix}\right)$$

$$= (\boxed{3},\boxed{6}) \otimes (\boxed{3},\boxed{6}) \otimes (\boxed{7},\boxed{6}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}\\ \boxed{2}&\boxed{7}&\boxed{7}&\boxed{9}\\ \boxed{3}&\boxed{8}\\ \boxed{8}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}&\boxed{4}\\ \boxed{3}&\boxed{3}&\boxed{4}&\boxed{5}\\ \boxed{4}&\boxed{6}\\ \boxed{6}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}\\ \boxed{3}&\boxed{3}\end{pmatrix} \otimes (\boxed{7},\boxed{6}) \otimes (\boxed{8},\boxed{6}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}\\ \boxed{2}&\boxed{7}&\boxed{7}&\boxed{9}\\ \boxed{3}\\ \boxed{8}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}&\boxed{4}\\ \boxed{3}&\boxed{3}&\boxed{4}&\boxed{5}\\ \boxed{4}\\ \boxed{6}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}&\boxed{6}\\ \boxed{3}&\boxed{3}&\boxed{7}\end{pmatrix} \otimes (\boxed{8},\boxed{6}) \otimes (\boxed{8},\boxed{6}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}\\ \boxed{2}&\boxed{7}&\boxed{7}&\boxed{9}\\ \boxed{3}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}&\boxed{4}\\ \boxed{3}&\boxed{3}&\boxed{4}&\boxed{5}\\ \boxed{4}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}\\ \boxed{3}&\boxed{3}&\boxed{7}&\boxed{8}&\boxed{8}\end{pmatrix} \otimes (\boxed{3},\boxed{5}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{1}&\boxed{2}\\ \boxed{2}&\boxed{7}&\boxed{9}\\ \boxed{7}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}&\boxed{4}\\ \boxed{3}&\boxed{3}&\boxed{4}\\ \boxed{4}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{5}\\ \boxed{3}&\boxed{3}&\boxed{7}&\boxed{8}&\boxed{8}&\boxed{3}\end{pmatrix} \otimes (\boxed{1},\boxed{4}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{2}\\ \boxed{2}&\boxed{7}&\boxed{9}\\ \boxed{7}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}\\ \boxed{3}&\boxed{3}&\boxed{4}\\ \boxed{4}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{5}&\boxed{4}\\ \boxed{3}&\boxed{3}&\boxed{7}&\boxed{8}&\boxed{8}&\boxed{3}&\boxed{1}\end{pmatrix} \otimes (\boxed{7},\boxed{4}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{2}\\ \boxed{2}&\boxed{9}\\ \boxed{7}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}\\ \boxed{3}&\boxed{3}\\ \boxed{4}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{5}&\boxed{4}&\boxed{4}\\ \boxed{3}&\boxed{3}&\boxed{7}&\boxed{8}&\boxed{8}&\boxed{3}&\boxed{1}&\boxed{7}\end{pmatrix} \otimes (\boxed{7},\boxed{4}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{2}\\ \boxed{2}&\boxed{9}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}\\ \boxed{3}&\boxed{3}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{5}&\boxed{4}&\boxed{4}&\boxed{4}\\ \boxed{3}&\boxed{3}&\boxed{7}&\boxed{8}&\boxed{8}&\boxed{3}&\boxed{1}&\boxed{7}&\boxed{7}\end{pmatrix} \otimes (\boxed{2},\boxed{3}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{2}\\ \boxed{9}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}\\ \boxed{3}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{5}&\boxed{4}&\boxed{4}&\boxed{4}&\boxed{3}\\ \boxed{3}&\boxed{3}&\boxed{7}&\boxed{8}&\boxed{8}&\boxed{3}&\boxed{1}&\boxed{7}&\boxed{7}&\boxed{2}\end{pmatrix} \otimes (\boxed{9},\boxed{3}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{1}&\boxed{2}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}&\boxed{2}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{5}&\boxed{4}&\boxed{4}&\boxed{4}&\boxed{3}&\boxed{3}\\ \boxed{3}&\boxed{3}&\boxed{7}&\boxed{8}&\boxed{8}&\boxed{3}&\boxed{1}&\boxed{7}&\boxed{7}&\boxed{2}&\boxed{9}\end{pmatrix} \otimes (\boxed{1},\boxed{2}) \otimes \left(\begin{matrix}\boxed{1}&\boxed{2}\end{matrix}, \begin{matrix}\boxed{1}&\boxed{2}\end{matrix}\right)$$

$$= \begin{pmatrix}\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{6}&\boxed{5}&\boxed{4}&\boxed{4}&\boxed{4}&\boxed{3}&\boxed{3}&\boxed{2}\\ \boxed{3}&\boxed{3}&\boxed{7}&\boxed{8}&\boxed{8}&\boxed{3}&\boxed{1}&\boxed{7}&\boxed{7}&\boxed{2}&\boxed{9}&\boxed{1}\end{pmatrix} \otimes (\boxed{1},\boxed{2}) \otimes (\boxed{2},\boxed{1})$$

22.3 Pieri rules

Recall that $\tilde{q}_r = \tilde{q}_r(x;t,u)$ is defined by the generating function

$$\prod_{i=1}^{n} \frac{1-tx_iz}{1-ux_iz} = \sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{q}_r z^r.$$

and that

$$h_r(x) = \tilde{q}_r(x;0,1), \qquad e_r(x) = \tilde{q}_r(x;-1,0), \qquad p_r(x) = \Big(\frac{1}{u-t}\tilde{q}_r(x;t,u)\Big)\Big|_{t=u}. \qquad \text{(utspec)}$$

Expanding, factor by factor, the product

$$\prod_{i=1}^{n} \frac{1-tx_iz}{1-ux_iz} = \Big(\frac{1}{1-ux_1z}\Big)\cdots\Big(\frac{1}{1-ux_nz}\Big)(1-tx_nz)\cdots(1-tx_1z)$$

gives

$$\tilde{q}_r(x;t,u) = \sum_{i_1\leq i_2\leq\cdots\leq i_k > i_{k+1}>\cdots>i_r} u^{k-1}(-t)^{r-k} x_{i_1}\cdots x_{i_k}x_{i_{k+1}}\cdots x_{i_r}. \qquad \text{(tildeqincdecB)}$$

FIGURE OUT THE CLEAN WAY TO DEAL WITH TH $(u-t)$ factor.

- A *vertical strip* is a skew shape λ/μ which has at most one box in each row.
- A *horizontal strip* is a skew shape λ/μ which has at most one box in each column.
- A *border strip* is a skew shape λ/μ which has at most one box in each diagonal.
- A *connected border strip* is a border strip λ/μ such that ??

PICTURES

Theorem 22.2. *(Pieri rules) Let μ be a partition. Then*

$$\tilde{q}_r s_\mu = \sum_{\lambda/\mu \text{ bbs length } r} (-t)^{\mathrm{ht}(\lambda/\mu)-\ell(\mu)} q^{\#\mathrm{cols}(\lambda/\mu)-\ell(\mu)} s_\lambda \qquad \left(\begin{matrix}\text{sum over vertical horizontal} \\ \text{strips } \lambda/\mu \text{ of length } r\end{matrix}\right)$$

$$h_r s_\mu = \sum_{\lambda/\mu \text{ hs length } r} s_\lambda \qquad (\text{sum over horizontal strips } \lambda/\mu \text{ of length } r)$$

$$e_r s_\mu = \sum_{\lambda/\mu \text{ vs length } r} s_\lambda \qquad (\text{sum over vertical strips } \lambda/\mu \text{ of length } r)$$

$$p_r s_\mu = \sum_{\lambda/\mu \text{ bs length } r} (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda \qquad (\text{sum over connected border strips } \lambda/\mu \text{ of length } r)$$

Proof idea. Let

$$UD_r = \left\{\text{sequences } (i_r,\ldots,i_1) \;\middle|\; \begin{matrix} i_1,\ldots,i_r\in\{1,\ldots n\} \text{ and} \\ i_r\leq\cdots\leq i_{k+1}>i_k>\cdots>i_1\end{matrix}\right\}.$$

Then RSK insertion gives a bijection

$$\begin{matrix} U_r(k)\times B_n(\mu) & \longleftrightarrow & \bigsqcup_{\lambda/\mu \text{ vhs length } r} B_n(\lambda) \\ ((i_r,\ldots,i_1),P) & \longmapsto & Q \end{matrix}$$

which results in the identity

$$\tilde{q}_r(x;t,u)s_\mu(x) = \sum_{\lambda/\mu \text{ vhs length } r} u^{\#\mathrm{cols}(\lambda/\mu)-1}(-t)^{\#\mathrm{rows}(\lambda/\mu)-1}s_\lambda(x).$$

The other identites are obtained from this one by specializing t and u as in (utspec). $\square$

22.4 Character tables and Kostka numbers

Let $n \in \mathbb{Z}_{>0}$ and let λ, μ be partitions of n.

Let λ' denote the conjugate partition to λ.

Define

$$\chi^\lambda_{t,u}(\mu) = \sum_{Q\in\hat{S}^\lambda_n} \mathrm{rwt}^\mu_{t,u}(Q), \qquad \text{where} \quad \hat{S}^\lambda_n = \{\text{standard tableaux } Q \text{ of shape } \lambda\}$$

and

$$\mathrm{rwt}^\mu_{t,u}(Q) = \prod_{\substack{j\in Q\\ j\notin J(\mu)}} f_\mu(j;Q), \qquad \text{with} \quad J(\mu) = \{\mu_1, \mu+\mu_2, \ldots, \mu_1+\cdots+\mu_\ell\}$$

and

$$f_\mu(j;Q) = \begin{cases} -t, & \text{if } j+1 \text{ is sw of } j \text{ in } Q, \\ 0, & \text{if } j+1 \notin J(\mu) \text{ and} \\ & \quad j+1 \text{ is ne of } j \text{ in } Q \text{ and} \\ & \quad j+2 \text{ is sw of } j+1 \text{ in } Q, \\ u, & \text{otherwise,} \end{cases} \qquad \left(\begin{array}{l} \text{pictorially the} \\ \text{three cases} \\ \text{are} \end{array} \left\{ \begin{array}{l} \begin{array}{|c|}\hline j \\ \hline j+1 \\ \hline \end{array} \\[2ex] \begin{array}{|c|c|}\hline j & j+1 \\ \hline j+2 \\ \cline{1-1} \end{array} \\[2ex] \begin{array}{|c|c|}\hline j & j+1 \\ \hline \end{array} \end{array} \right. \right)$$

where

a box (r,c) is southwest (sw) of a box (a,b) if $r > a$ and $c \le b$, and
a box (r,c) is northeast (ne) of a box (a,b) if $r \le a$ and $c > b$.

Define

$$\chi^\lambda_{H_n}(\mu) = \chi^\lambda_{q,1}(\mu), \qquad \chi^\lambda_{S_n}(\mu) = \chi^\lambda_{1,1}(\mu) \qquad K_{\lambda\mu} = \chi_{1,0}(\mu), \qquad K_{\lambda'\mu'} = \chi_{0,-1}(\mu).$$

Corollary 22.3. *Let $n \in \mathbb{Z}_{>0}$ and let μ be a partition of n. Then*

$$\tilde{q}_\mu(x;u,t) = \sum_{\lambda\in\mathbb{Y}_n} \chi^\lambda_{u,t}(\mu)s_\lambda(x), \qquad \tilde{q}_\mu(q,1) = \sum_{\lambda\in\mathbb{Y}_n} \chi^\lambda_{H_n}(\mu)s_\lambda. \qquad p_\mu = \sum_{\lambda\in\mathbb{Y}_n} \chi^\lambda_{S_n}(\mu)s_\lambda,$$

$$h_\mu = \sum_{\lambda\in\mathbb{Y}_n} K_{\lambda\mu}s_\lambda, \qquad \textit{and} \qquad e_\mu = \sum_{\lambda\in\mathbb{Y}_n} K_{\lambda'\mu'}s_\lambda.$$

Since a SSYT is a sequence of horizontal strips then

$$K_{\lambda\mu} = \mathrm{Card}(B(\lambda)_\mu) = \chi^\lambda_{1,0}(\mu)$$

so that $K_{\lambda\mu}$ is the number of SSYT of shape λ and weight μ. The matrices

$$\chi_{S_n} = (\chi^\lambda_{S_n}(\mu))_{\lambda,\mu\in\mathbb{Y}_n} \qquad \text{and} \qquad \chi_{H_n(q)} = (\chi^\lambda_{H_n(q)}(\mu))_{\lambda,\mu\in\mathbb{Y}_n}$$

are the *character tables of the symmetric group and the Hecke algebra*, respectively.

Examples: The sets of standard tableaux for partitions with ≤ 4 boxes are:

$$S_1^{\ydiagram{1}} = \{\ytableaushort{1}\}, \qquad S_2^{\ydiagram{2}} = \{\ytableaushort{12}\}, \qquad S_2^{\ydiagram{1,1}} = \{\ytableaushort{1,2}\},$$

$$S_3^{\ydiagram{3}} = \{\ytableaushort{123}\}, \qquad S_3^{\ydiagram{2,1}} = \{\ytableaushort{12,3}, \ytableaushort{13,2}\}, \qquad S_3^{\ydiagram{1,1,1}} = \{\ytableaushort{1,2,3}\},$$

$$S_4^{\ydiagram{4}} = \{\ytableaushort{1234}\}, \qquad S_4^{\ydiagram{3,1}} = \{\ytableaushort{123,4}, \ytableaushort{124,3}, \ytableaushort{134,2}\}, \qquad S_4^{\ydiagram{2,2}} = \{\ytableaushort{12,34}, \ytableaushort{13,24}\},$$

$$S_4^{\ydiagram{2,1,1}} = \{\ytableaushort{12,3,4}, \ytableaushort{13,2,4}, \ytableaushort{14,2,3}\}, \qquad S_4^{\ydiagram{1,1,1,1}} = \{\ytableaushort{1,2,3,4}\},$$

The matrices $\chi_{u,t}$ for $n \in \{2,3,4\}$ are

$$\chi_{u,t} = \begin{array}{c|cc} \lambda/\mu & (2) & (1^2) \\ \hline (2) & u & 1 \\ (1^2) & -t & 1 \end{array} \qquad \chi_{u.t} = \begin{array}{c|ccc} \lambda/\mu & (3) & (21) & (1^3) \\ \hline (3) & u^2 & u & 1 \\ (21) & 0+u(-t) & u+(-t) & 1+1 \\ (1^3) & (-t)^2 & -t & 1 \end{array}$$

$$\chi_{u,t} = \begin{array}{c|ccccc} \lambda/\mu & (4) & (31) & (22) & (21^2) & (1^4) \\ \hline (4) & 1 & 1 & 1 & 1 & 1 \\ (31) & 0+0+(-t) & 0+(-t)+1 & (-t)+1+(-t) & (-t)+1+1 & 1+1+1 \\ (22) & 0+0 & 0+(-t) & (-t)^2+1 & (-t)+1 & 1+1 \\ (21^2) & 0+0+(-t)^2 & (-t)^2+0+(-t) & (-t)+(-t)^2+(-t) & (-t)+(-t)+1 & 1+1+1 \\ (1^4) & (-t)^3 & (-t)^2 & (-t)^2 & (-t) & 1 \end{array}$$

The matrices K for $n \in \{2,3,4\}$ are

$$K = K(0,1) = \begin{array}{c|cc} \lambda\backslash\mu & (2) & (1^2) \\ \hline (2) & 1 & 1 \\ (1^2) & 0 & 1 \end{array} \qquad K = K(0,1) = \begin{array}{c|ccc} \lambda\backslash\mu & (3) & (21) & (1^3) \\ \hline (3) & 1 & 1 & 1 \\ (21) & 0 & 1 & 2 \\ (1^3) & 0 & 0 & 1 \end{array}$$

$$K = K(0,1) = \begin{array}{c|ccccc} \mu/\lambda & (4) & (31) & (22) & (21^2) & (1^4) \\ \hline (4) & 1 & 1 & 1 & 1 & 1 \\ (31) & 0 & 1 & 1 & 2 & 3 \\ (22) & 0 & 0 & 1 & 1 & 2 \\ (21^2) & 0 & 0 & 0 & 1 & 3 \\ (1^4) & 0 & 0 & 0 & 0 & 1 \end{array}$$

Example 22.2. *The following is an example of the RSK insertion used in the proof of Theorem 22.2.*

$$\begin{pmatrix} \boxed{27} & \boxed{26} & \boxed{25} & \boxed{24} & \boxed{23} & \boxed{22} & \boxed{21} & \boxed{20} & \boxed{19} & \boxed{18} & \boxed{17} & \boxed{16} & \boxed{15} \\ \boxed{1} & \boxed{3} & \boxed{3} & \boxed{5} & \boxed{5} & \boxed{5} & \boxed{7} & \boxed{7} & \boxed{7} & \boxed{6} & \boxed{4} & \boxed{3} & \boxed{1} \end{pmatrix} \otimes \left(\begin{ytableau} 1 & 1 & 1 & 2 & 7 & 9 \\ 2 & 3 & 3 & 7 \\ 3 & 8 & 8 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \\ 11 & 12 & 13 \\ 14 \end{ytableau} \right)$$

$$= \begin{pmatrix} \boxed{27} & \boxed{26} & \boxed{25} & \boxed{24} & \boxed{23} & \boxed{22} & \boxed{21} & \boxed{20} \\ \boxed{1} & \boxed{3} & \boxed{3} & \boxed{5} & \boxed{5} & \boxed{5} & \boxed{7} & \boxed{7} \end{pmatrix} \otimes \left(\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 7 & 9 \\ 2 & 3 & 3 & 3 & 7 \\ 3 & 7 & 8 & 8 \\ 4 \\ 6 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & {\color{red}15} \\ 7 & 8 & 9 & 10 & {\color{red}16} \\ 11 & 12 & 13 & {\color{red}17} \\ 14 \\ {\color{red}18} \\ {\color{red}19} \end{ytableau} \right)$$

$$= \begin{pmatrix} \boxed{27} & \boxed{26} & \boxed{25} & \boxed{24} & \boxed{23} & \boxed{22} & \boxed{21} \\ \boxed{1} & \boxed{3} & \boxed{3} & \boxed{5} & \boxed{5} & \boxed{5} & \boxed{7} \end{pmatrix} \otimes \left(\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 7 & 9 \\ 2 & 3 & 3 & 3 & 7 \\ 3 & 7 & 7 & 8 & 8 \\ 4 \\ 6 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & 15 \\ 7 & 8 & 9 & 10 & 16 \\ 11 & 12 & 13 & 17 & 20 \\ 14 \\ 18 \\ 19 \end{ytableau} \right)$$

$$= \begin{pmatrix} \boxed{27} & \boxed{26} & \boxed{25} & \boxed{24} & \boxed{23} & \boxed{22} \\ \boxed{1} & \boxed{3} & \boxed{3} & \boxed{5} & \boxed{5} & \boxed{5} \end{pmatrix} \otimes \left(\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 7 & 9 \\ 2 & 3 & 3 & 3 & 7 & 8 \\ 3 & 7 & 7 & 7 & 8 \\ 4 \\ 6 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & 15 \\ 7 & 8 & 9 & 10 & 16 & 21 \\ 11 & 12 & 13 & 17 & 20 \\ 14 \\ 18 \\ 19 \end{ytableau} \right)$$

$$= \begin{pmatrix} \boxed{27} & \boxed{26} & \boxed{25} & \boxed{24} & \boxed{23} \\ \boxed{1} & \boxed{3} & \boxed{3} & \boxed{5} & \boxed{5} \end{pmatrix} \otimes \left(\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 7 & 7 & 9 \\ 2 & 3 & 3 & 3 & 7 & 8 \\ 3 & 6 & 7 & 7 & 8 \\ 4 \\ 5 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & 15 & 22 \\ 7 & 8 & 9 & 10 & 16 & 21 \\ 11 & 12 & 13 & 17 & 20 \\ 14 \\ 18 \\ 19 \end{ytableau} \right)$$

$$= \begin{pmatrix} \boxed{27} & \boxed{26} & \boxed{25} & \boxed{24} \\ \boxed{1} & \boxed{3} & \boxed{3} & \boxed{5} \end{pmatrix} \otimes \left(\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 7 & 7 & 7 & 9 \\ 2 & 3 & 3 & 3 & 7 & 8 \\ 3 & 5 & 6 & 7 & 8 \\ 4 \\ 5 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & 15 & 22 & 23 \\ 7 & 8 & 9 & 10 & 16 & 21 \\ 11 & 12 & 13 & 17 & 20 \\ 14 \\ 18 \\ 19 \end{ytableau} \right)$$

$$= \begin{pmatrix} \boxed{27} & \boxed{26} & \boxed{25} \\ \boxed{1} & \boxed{3} & \boxed{3} \end{pmatrix} \otimes \left(\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 7 & 7 & 7 & 7 & 9 \\ 2 & 3 & 3 & 3 & 7 & 8 \\ 3 & 5 & 5 & 6 & 8 \\ 4 \\ 5 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & 15 & 22 & 23 & 24 \\ 7 & 8 & 9 & 10 & 16 & 21 \\ 11 & 12 & 13 & 17 & 20 \\ 14 \\ 18 \\ 19 \end{ytableau} \right)$$

$$= \begin{pmatrix} \boxed{27} & \boxed{26} \\ \boxed{1} & \boxed{3} \end{pmatrix} \otimes \left(\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 7 & 7 & 7 & 7 & 7 & 9 \\ 2 & 3 & 3 & 3 & 3 & 8 \\ 3 & 5 & 5 & 6 & 8 \\ 4 \\ 5 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & 15 & 22 & 23 & 24 & 25 \\ 7 & 8 & 9 & 10 & 16 & 21 \\ 11 & 12 & 13 & 17 & 20 \\ 14 \\ 18 \\ 19 \end{ytableau} \right)$$

$$= \begin{pmatrix} \boxed{27} \\ \boxed{1} \end{pmatrix} \otimes \left(\begin{ytableau} 1 & 1 & 1 & 1 & 2 & 3 & 7 & 7 & 7 & 7 & 7 & 9 \\ 2 & 3 & 3 & 3 & 3 & 8 \\ 3 & 5 & 5 & 6 & 8 \\ 4 \\ 5 \\ 7 \end{ytableau}, \begin{ytableau} 1 & 2 & 3 & 4 & 5 & 6 & 15 & 22 & 23 & 24 & 25 & 26 \\ 7 & 8 & 9 & 10 & 16 & 21 \\ 11 & 12 & 13 & 17 & 20 \\ 14 \\ 18 \\ 19 \end{ytableau} \right)$$

giving

$$\left(\begin{array}{l}
\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{7}\boxed{7}\boxed{7}\boxed{7}\boxed{7}\boxed{9} \\
\boxed{2}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{8} \\
\boxed{3}\boxed{5}\boxed{5}\boxed{6}\boxed{8} \\
\boxed{4} \\
\boxed{5} \\
\boxed{7}
\end{array},\
\begin{array}{l}
\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{\color{red}{15}}\boxed{\color{blue}{22}}\boxed{\color{blue}{23}}\boxed{\color{blue}{24}}\boxed{\color{blue}{25}}\boxed{\color{blue}{26}}\boxed{\color{blue}{27}} \\
\boxed{7}\boxed{8}\boxed{9}\boxed{10}\boxed{\color{red}{16}}\boxed{\color{blue}{21}} \\
\boxed{11}\boxed{12}\boxed{13}\boxed{\color{red}{17}}\boxed{\color{blue}{20}} \\
\boxed{14} \\
\boxed{\color{red}{18}} \\
\boxed{\color{red}{19}}
\end{array}\right)$$

$\square$

Example 22.3. *The following is an example of the RSK insertion used in the proof of Theorem 22.2.*

$$\begin{aligned}
&\begin{pmatrix}
\boxed{13} & \boxed{12} & \boxed{11} & \boxed{10} & \boxed{9} & \boxed{8} & \boxed{7} & \boxed{6} & \boxed{5} & \boxed{4} & \boxed{3} & \boxed{2} & \boxed{1} \\
\boxed{1} & \boxed{3} & \boxed{3} & \boxed{5} & \boxed{5} & \boxed{5} & \boxed{7} & \boxed{7} & \boxed{7} & \boxed{6} & \boxed{4} & \boxed{3} & \boxed{1}
\end{pmatrix} \\
&= \begin{pmatrix}
\boxed{13} & \boxed{12} & \boxed{11} & \boxed{10} & \boxed{9} & \boxed{8} & \boxed{7} & \boxed{6} \\
\boxed{1} & \boxed{3} & \boxed{3} & \boxed{5} & \boxed{5} & \boxed{5} & \boxed{7} & \boxed{7}
\end{pmatrix} \otimes \left(\begin{array}{l}\boxed{1}\\\boxed{3}\\\boxed{4}\\\boxed{6}\\\boxed{7}\end{array}, \begin{array}{l}\boxed{1}\\\boxed{2}\\\boxed{3}\\\boxed{4}\\\boxed{5}\end{array}\right) \\
&= \left(\begin{array}{l}
\boxed{1}\boxed{1}\boxed{3}\boxed{3}\boxed{5}\boxed{5}\boxed{7}\boxed{7} \\
\boxed{3} \\
\boxed{4} \\
\boxed{6} \\
\boxed{7}
\end{array},\
\begin{array}{l}
\boxed{1}\boxed{6}\boxed{7}\boxed{8}\boxed{9}\boxed{10}\boxed{11}\boxed{12}\boxed{13} \\
\boxed{2} \\
\boxed{3} \\
\boxed{4} \\
\boxed{5}
\end{array}\right)
\end{aligned}$$

$\square$

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