16 The Modular group

By definition

$$SL_2(\mathbb{Z}) = \ker(\det), \qquad \text{where} \quad \det\colon GL_2(\mathbb{Z}) \to \mathbb{Z}^\times = \{\pm 1\}$$

so that there is an exact sequence

$$\{1\} \to SL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}) \xrightarrow{\det} \{\pm 1\} \to \{1\}$$

and $SL_2(\mathbb{Z})$ is index 2 in $GL_2(\mathbb{Z})$ with

$$GL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \sqcup SL_2(\mathbb{Z}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$Z(GL_2(\mathbb{Z})) = Z(SL_2(\mathbb{Z})) = \{\pm 1\}$$

and $PGL_2(\mathbb{Z})$ and $SL_2(\mathbb{Z})$ are defined by

$$PGL_2(\mathbb{Z}) = \frac{GL_2(\mathbb{Z})}{Z(GL_2(\mathbb{Z}))} \qquad \text{and} \qquad PSL_2(\mathbb{Z}) = \frac{SL_2(\mathbb{Z})}{Z(SL_2(\mathbb{Z}))}$$

and $PSL_2(\mathbb{Z})$ is index 2 in $PGL_2(\mathbb{Z})$ with

$$PGL_2(\mathbb{Z}) = PSL_2(\mathbb{Z}) \sqcup PSL_2(\mathbb{Z}) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Proposition 16.1. *The following are equivalent presentations of $GL_2(\mathbb{Z})$.*

(a) Generators: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $ad - bc \in \mathbb{Z}^\times$*,*

Relations:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}$$

(b) Generators: γ_1, γ_2, e

Relations

$$\gamma_1^2 \in Z(GL_2(\mathbb{Z})), \quad \gamma_1^4 = 1, \quad \gamma_2^3 = \gamma_1^2, \tag{b1}$$

$$e^2 = 1, \quad e\gamma_1 e^{-1} = \gamma_1^{-1}, \quad e\gamma_2 e^{-1} = \gamma_1\gamma_2^{-1}\gamma_1^{-1}. \tag{b2}$$

(c) Generators: σ_1, σ_2, e

Relations:

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \qquad (\sigma_1\sigma_2\sigma_1)^2 \in Z(GL_2(\mathbb{Z})), \qquad (\sigma_1\sigma_2\sigma_1)^4 = 1, \tag{c1}$$

$$e^2 = 1, \quad e\sigma_1 e^{-1} = \sigma_1^{-1}, \quad e\sigma_2 e^{-1} = \sigma_2^{-1} \tag{c2}$$

(d) Generators: $x_{12}(1), x_{21}(1), n_1, e,$

Relations:

$$x_{12}(1)x_{21}(-1)x_{12}(1) = n_1, \qquad n_1^2 \in Z(GL_2(\mathbb{Z})), \qquad n_1^4 = 1, \tag{d1}$$

$$n_1 x_{12}(1) n_1^{-1} = x_{21}(-1), \qquad n_1 x_{21}(1) n_1^{-1} = x_{12}(-1), \tag{d2}$$

$$e^2 = 1, \quad e x_{12}(1) e^{-1} = x_{12}(-1), \quad e x_{21}(1) e^{-1} = x_{21}(-1), \quad e n_1 e^{-1} = n_1^{-1}, \tag{d3}$$

where we use the notation

$$x_{12}(a) = x_{12}(1)^a \quad \textit{and} \quad x_{21}(a) = x_{21}(1)^a, \quad \textit{for } a \in \mathbb{Z}.$$

Proof. Generators (b) in terms of generators (a), generators (c) and generators (d):

$$\begin{aligned}\gamma_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma_1\sigma_2\sigma_1 = n_1, \\ \gamma_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \sigma_1\sigma_2 = x_{21}(1)n_1,\end{aligned} \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e = e.$$

Generators (c) in terms of generators (a), generators (b) and generators (d):

$$\begin{aligned}\sigma_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \gamma_2^{-1}\gamma_1 = x_{12}(1), \\ \sigma_2 &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \gamma_1\gamma_2^{-1} = x_{21}(-1),\end{aligned} \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e = e. \tag{16.1}$$

Generators (d) in terms of generators (a), generators (b) and generators (c):

$$x_{12}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \gamma_2^{-1}\gamma_1 = \sigma_1, \quad x_{21}(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \gamma_2\gamma_1^{-1} = \sigma_2^{-1},$$

$$n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \gamma_1 = \sigma_1\sigma_2\sigma_1, \quad e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e = e.$$

Relations (c) from relations (b): The computations

$$\sigma_1\sigma_2\sigma_1 = \gamma_2^{-1}\gamma_1\gamma_1\gamma_2^{-1}\gamma_2^{-1}\gamma_1 = \gamma_2^{-3}\gamma_1^2\gamma_1 = \gamma_1^{-2}\gamma_1^2\gamma_1 = \gamma_1,$$

$$\sigma_2\sigma_1\sigma_2 = \gamma_1\gamma_2^{-1}\gamma_2^{-1}\gamma_1\gamma_1\gamma_2^{-1} = \gamma_1^3\gamma_2^{-2}\gamma_2^{-1} = \gamma_1^3\gamma_2^{-3} = \gamma_1^3\gamma_1^{-2} = \gamma_1,$$

give

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \qquad (\sigma_1\sigma_2\sigma_1)^2 = \gamma_1^2 \in Z(GL_2(\mathbb{Z})), \qquad (\sigma_1\sigma_2\sigma_1)^4 = \gamma_1^4 = 1,$$
$$e^2 = e^2 = 1, \qquad e\sigma_1e^{-1} = e\gamma_2^{-1}\gamma_1e^{-1} = \gamma_1\gamma_2\gamma_1^{-1}\gamma_1^{-1} = \gamma_1^{-2}\gamma_1\gamma_2 = \gamma_1^{-1}\gamma_2 = \sigma_1^{-1},$$
$$e\sigma_2e^{-1} = e\gamma_1\gamma_2^{-1}e^{-1} = \gamma_1^{-1}\gamma_1\gamma_2\gamma_1^{-1} = \gamma_2\gamma_1^{-1} = \sigma_2^{-1}.$$

Relations (b) from relations (c): Compute

$$\gamma_1^2 = (\sigma_1\sigma_2\sigma_1)^2 \in Z(GL_2(\mathbb{Z})), \qquad \gamma_1^4 = (\sigma_1\sigma_2\sigma_1)^4 = 1,$$

$$\gamma_2^3 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1 = (\sigma_1\sigma_2\sigma_1)^2 = \gamma_1^2,$$
$$e^2 = e^2 = 1, \qquad e\gamma_1e^{-1} = e\sigma_1\sigma_2\sigma_1e^{-1} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} = \gamma_1^{-1},$$

and

$$\begin{aligned}e\gamma_2e^{-1} &= e\sigma_1\sigma_2e^{-1} = \sigma_1^{-1}\sigma_2^{-1} = \sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} \\ &= \sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} = \sigma_2\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}) \\ &= (\sigma_1\sigma_2\sigma_1)\sigma_2^{-1}\sigma_1^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}) = \gamma_1\gamma_2^{-1}\gamma_1^{-1}.\end{aligned}$$

Relations (b) from relations (d): Compute

$$\gamma_1^2 = n_1^2 \in Z(GL_2(\mathbb{Z})), \qquad \gamma_1^4 = n_1^4 = 1, \qquad e^2 = e^2 = 1, \qquad e\gamma_1e^{-1} = en_1e^{-1} = n_1^{-1},$$

$$\begin{aligned}\gamma_2^3 &= x_{21}(1)n_1x_{21}(1)n_1x_{21}(1)n_1 = n_1x_{12}(-1)n_1x_{12}(-1)n_1x_{12}(-1)\\ &= n_1^2x_{21}(1)n_1x_{21}(1)x_{12}(-1) = n_1^3x_{12}(-1)x_{21}(1)x_{12}(-1) = n_1^3n_1^{-1} = n_1^2 = \gamma_1^2,\end{aligned}$$

and

$$e\gamma_2e^{-1} = ex_{21}(1)n_1e^{-1} = x_{21}(-1)n_1^{-1} = n_1n_1^{-1}x_{21}(-1)n_1^{-1} = \gamma_1\gamma_2^{-1}\gamma_1^{-1}.$$

Relations (d) from relations (b): Compute

$$x_{12}(1)x_{21}(-1)x_{12}(1) = \gamma_2^{-1}\gamma_1\gamma_1\gamma_2^{-1}\gamma_2^{-1}\gamma_1 = \gamma_1^2\gamma_2^{-3}\gamma_1 = \gamma_1 = n_1,$$

$$n_1^2 = \gamma_1^2 \in Z(GL_2(\mathbb{Z})), \qquad n_1^4 = \gamma_1^4 = 1, \qquad e^2 = e^2 = 1,$$

$$n_1x_{12}(1)n_1^{-1} = \gamma_1\gamma_2^{-1}\gamma_1\gamma_1^{-1} = \gamma_1\gamma_2^{-1} = x_{21}(1)^{-1} = x_{21}(-1),$$

$$n_1x_{21}(1)n_1^{-1} = \gamma_1\gamma_2\gamma_1^{-1}\gamma_1^{-1} = \gamma_1^{-2}\gamma_1\gamma_2 = \gamma_1^{-1}\gamma_2 = x_{12}(1)^{-1} = x_{12}(-1),$$

$$ex_{21}(1)e^{-1} = e\gamma_2^{-1}\gamma_1e^{-1} = \gamma_1\gamma_2\gamma_1^{-1}\gamma_1^{-1} = \gamma_1^{-2}\gamma_1\gamma_2 = \gamma_1^{-1}\gamma_2 = x_{12}(1)^{-1} = x_{12}(-1),$$

$$ex_{21}(1)e^{-1} = e\gamma_2\gamma_1^{-1}e^{-1} = \gamma_1\gamma_2^{-1}\gamma_1^{-1}\gamma_1 = \gamma_1\gamma_2^{-1} = x_{21}(1)^{-1} = x_{21}(-1),$$

and

$$en_1e^{-1} = e\gamma_1e^{-1} = \gamma_1^{-1} = n_1^{-1}.$$

Relations (d) from relations (a): Compute

$$x_{12}(1)x_{21}(-1)x_{12}(1) = \begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix} = \begin{pmatrix}0&1\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = n_1,$$

$$n_1^2 = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&-1\end{pmatrix} \in Z(GL_2(\mathbb{Z})), \qquad e^2 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}^2 = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1,$$

$$n_1^4 = (n_1^2)^2 = \begin{pmatrix}-1&0\\0&-1\end{pmatrix}^2 = \begin{pmatrix}1&0\\0&1\end{pmatrix} = 1,$$

$$n_1x_{12}(1)n_1^{-1} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\-1&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\-1&1\end{pmatrix} = x_{21}(-1),$$

$$n_1x_{21}(1)n_1^{-1} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}1&1\\-1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}1&-1\\0&1\end{pmatrix} = x_{12}(-1),$$

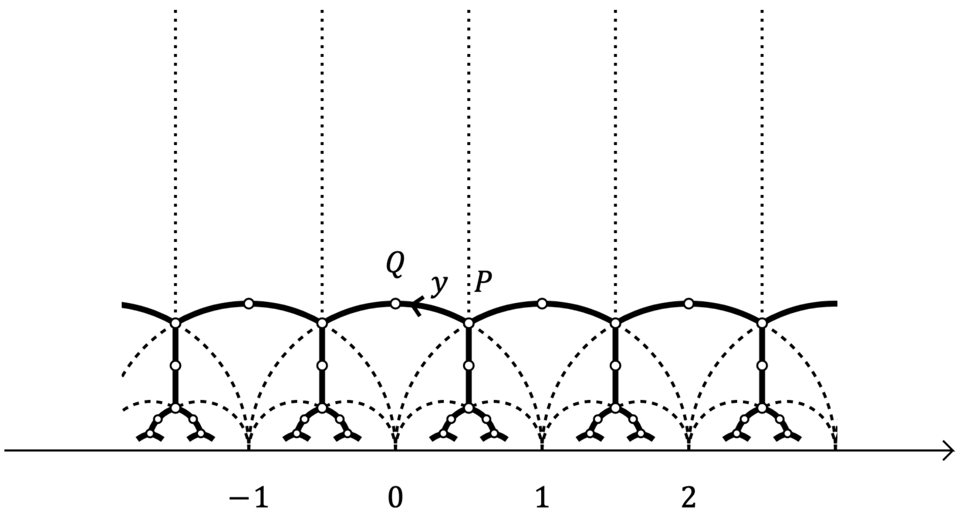
$$ex_{12}(1)e^{-1} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&1\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&-1\\0&1\end{pmatrix} = x_{12}(-1),$$

$$ex_{21}(1)e^{-1} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\-1&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\-1&1\end{pmatrix} = x_{21}(-1),$$

and

$$en_1e^{-1} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix} = n_1^{-1}.$$

Relations generators? (a) from relations generators? (b): Organize this according to the 3-valent tree (see [Serre, 4.2(c) and Theorem 6]).



Mark one edge as the identity edge A, and other edges as Ag, so that

$$PSL_2(\mathbb{Z}) \longrightarrow \{\text{edges of tree}\} g \longmapsto Ag \qquad \text{is a bijection.}$$

There are two parts of the graph, a left side and a ride side, separated by the vertex at i, with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on the right side of i and $\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the left side of i. The segment between $e^{i\pi/3}$ and $e^{-i\pi/3}$ always has image with edges labeled by g and $g\gamma_1$ for some $g \in PSL_2(\mathbb{Z})$. If this edge is on the right side of the graph then, together with its partner edge on the left side of the graph, these edges represent the four elements

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g\gamma_1 = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}, \quad \gamma_1 g = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}, \quad \gamma_1 g \gamma_1 = \begin{pmatrix} -d & c \\ b & -a \end{pmatrix}.$$

In this way we can restrict attention to the right side of the graph and the "right side" of the double edges. The steps from such an edge to the neighboring edges are

$$\gamma_2\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = x_{21}(1),$$

$$\gamma_2^{-1}\gamma_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = x_{12}(1),$$

as elements of $PSL_2(\mathbb{Z})$. So every element on the "right-right" portion of the graph should have a unique expression as a word in the letters $x_{12}(1)$ and $x_{21}(1)$ (these two elements should form a free monoid inside $PSL_2(\mathbb{Z})$). Any product of $x_{12}(1)$ and $x_{21}(1)$ has all positive entries.

Assume

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with } a, b, c, d \in \mathbb{Z}_{\geq 0}.$$

Produce a geodesic (minimal length) path back to the identity as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} a-b & b \\ c-d & d \end{pmatrix}, \qquad \text{when } a > b,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-a \\ c & d-c \end{pmatrix}, \qquad \text{when } b \geq a.$$

Let us think about when $a > b$ and $c < d$ (which could cause a negative entry). Then $a \geq b+1$ and $d \geq c+1$ gives

$$ad \geq (b+1)(c+1) = bc + b + c + 1, \quad \text{and since} \quad ad - bc = 1, \quad \text{then} \quad b = c = 0,$$

which also forces $a = d = 1$. Thus, if the starting point is not the identity and $a > b$, then $c \geq d$ and the next matrix in the sequence also has positive entries. A similar argument handles the case $b > a$. When $b = a$ then $a(d-c) = ad - ac = 1$ forces $d - c = 1$ and $a = 1$. □

Theorem 16.2.

(a) *$SL_2(\mathbb{Z}_{\geq 0}$ is the free monoid on two generators* $x_{12}(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $x_{21}(1) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(b) *$SL_2(\mathbb{Z})$ is 8 copies of $SL_2(\mathbb{Z}_{\geq 0})$:*

$$\begin{aligned} SL_2(\mathbb{Z}) = {} & SL_2(\mathbb{Z}_{\geq 0}) \sqcup SL_2(\mathbb{Z}_{\geq 0})\gamma_1 \sqcup \gamma_1 SL_2(\mathbb{Z}_{\geq 0}) \sqcup \gamma_1 SL_2(\mathbb{Z}_{\geq 0})\gamma_1 \\ & \sqcup (-1)SL_2(\mathbb{Z}_{\geq 0}) \sqcup (-1)SL_2(\mathbb{Z}_{\geq 0})\gamma_1 \sqcup (-1)\gamma_1 SL_2(\mathbb{Z}_{\geq 0}) \sqcup (-1)\gamma_1 SL_2(\mathbb{Z}_{\geq 0})\gamma_1, \end{aligned}$$

where $\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and* $(-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) *$GL_2(\mathbb{Z})$ is 2 copies of $SL_2(\mathbb{Z})$:* $GL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \sqcup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} SL_2(\mathbb{Z})$.

16.1 $GL_2(\mathbb{Z})$

The group $GL_2(\mathbb{Z})$ is

$$GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{Z} \text{ and } ad - bc \in \mathbb{Z}^\times \right\}, \qquad \text{where } \mathbb{Z}^\times = \{1, -1\}$$

is the group of invertible elements of $\mathbb{Z}$. Define

$$\sigma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

so that $\gamma_1 = \sigma_1\sigma_2\sigma_1$ and $\gamma_2 = \sigma_1\sigma_2$ and $s = e\gamma_1$.

Proposition 16.3.

(a) *The group $GL_2(\mathbb{Z})$ is presented by generators σ_1, σ_2 and s with relations*

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \qquad (\sigma_1\sigma_2\sigma_1)^2 \in Z(GL_2(\mathbb{Z})), \qquad (\sigma_1\sigma_2\sigma_1)^4 = 1,$$
$$s^2 = 1, \qquad s\sigma_1 = \sigma_2^{-1}s, \qquad s\sigma_2 = \sigma_1^{-1}s.$$

(b) The group $GL_2(\mathbb{Z})$ is presented by generators γ_1, γ_2 and e with relations

$$\gamma_1^4 = 1, \qquad \gamma_2^3 = \gamma_1^2, \qquad \gamma_1^2 \in Z(GL_2(\mathbb{Z})),$$
$$s^2 = 1, \qquad s\gamma_1 = \gamma_1^{-1}s, \qquad s\gamma_2 = \gamma_2^{-1}s.$$

Remark 16.4. The braid group B_3 on 3 strands is presented by generators σ_1, σ_2 with relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Then

$$Z(B_3) = \{(\sigma_1\sigma_2\sigma_1)^k \mid k \in \mathbb{Z}\}, \quad \text{and} \quad \sigma_1 = PICTURE, \qquad \sigma_2 = PICTURE,$$

in a familiar braid representation of B_3. These pictures are a realization of the isomorphism between B_3 and the fundamental groups of the configuration space of 3 distinct points on a plane,

$$B_3 \cong \pi_1\Big(\frac{\mathbb{C}^3 - (H_{12} \cup H_{13} \cup H_{23})}{S_3}\Big), \qquad \text{where} \qquad \begin{array}{l} H_{12} = \{(a_1, a_1, a_3) \in \mathbb{C}^3\}, \\ H_{13} = \{(a_1, a_2, a_1) \in \mathbb{C}^3\}, \\ H_{23} = \{(a_1, a_2, a_2) \in \mathbb{C}^3\}, \end{array}$$

and the group S_3 acts on $\mathbb{C}^3 - (H_{12} \cup H_{13} \cup H_{23})$ by permuting the coordinates.

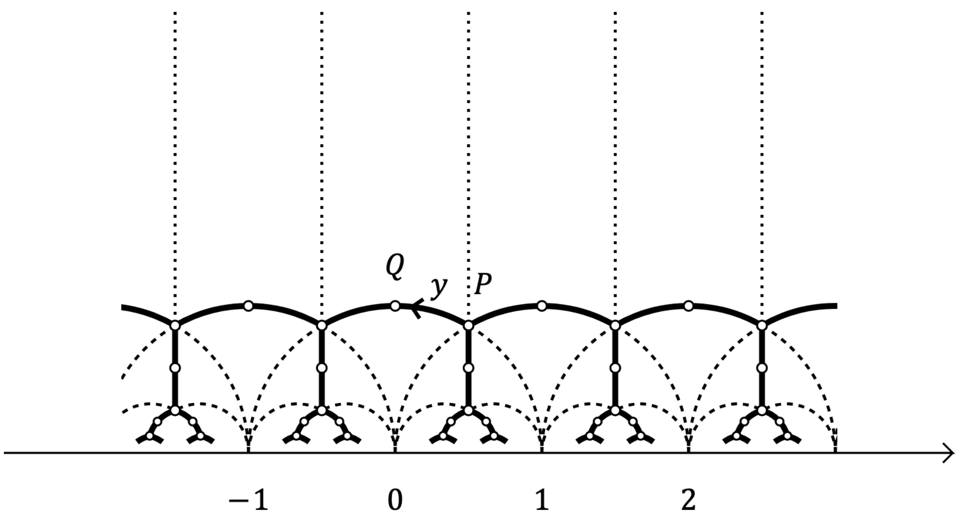
Remark 16.5. The group $SL_2(\mathbb{Z})$ is the subgroup of $GL_2(\mathbb{Z})$ generated by γ_1 and γ_2. The group $SL_2(\mathbb{Z})$ acts on the upper half plane

$$\mathfrak{S}_1 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \in \mathbb{R}_{>0}\}, \qquad \text{by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

The stabilizer of the points i and $e^{i\pi/3}$ are

$$\mathrm{Stab}(i) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle = \langle \gamma_1 \rangle \cong \mathbb{Z}/4\mathbb{Z}, \quad \text{and} \quad \mathrm{Stab}(e^{i\pi/3}) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle = \langle \gamma_2 \rangle \cong \mathbb{Z}/6\mathbb{Z},$$

and the infinite three valent tree is generated by the action of $SL_2(\mathbb{Z})$ on the arc of the unit circle connecting i and $e^{i\pi/3}$. The arc of the unit circle connecting i and $e^{i\pi/3}$ (in fact every element of $\mathfrak{S}_1$) is stabilized by $-1 \in SL_2(\mathbb{Z})$.



(uHPpic)

(see [Serre, 4.2(c) and Theorem 6]).

Remark 16.6. For a general commutative ring R there is an exact sequence

$$\{1\} \to SL_n(R) \to GL_n(R) \xrightarrow{\det} R^\times \to \{1\}, \qquad \text{where } R^\times = \{r \in R \mid r^{-1} \in R\},$$

and

$$GL_n(R) = \bigsqcup_{r \in R^\times} SL_n(R) h_n(r), \qquad \text{where} \quad h_n(r) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & r \end{pmatrix}.$$

For a general commutative ring R

$$Z(GL_n(R)) = R^\times \qquad \text{and} \qquad Z(SL_n(R)) = \{r \in R \mid r^n = 1\}.$$

□

Remark 16.7. The subgroup $SL_2(\mathbb{Z})$ is generated by the sets

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\}, \qquad \{\gamma_1, \gamma_2\}, \qquad \{\sigma_1, \sigma_2\}, \qquad \text{and} \qquad \{x_{12}(1), x_{21}(1), n_1\}.$$

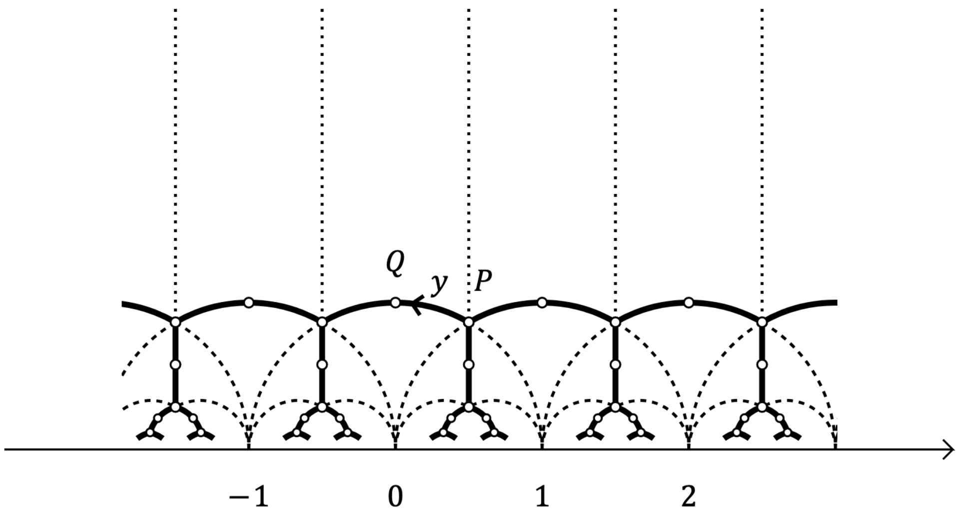
Remark 16.8. The group $SL_2(\mathbb{Z})$ acts on the upper half plane

$$\mathfrak{S}_1 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \in \mathbb{R}_{>0}\}, \qquad \text{by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}.$$

The stabilizer of the points i and $e^{i\pi/3}$ are

$$\mathrm{Stab}(i) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle = \langle \gamma_1 \rangle \cong \mathbb{Z}/4\mathbb{Z}, \quad \text{and} \quad \mathrm{Stab}(e^{i\pi/3}) = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle = \langle \gamma_2 \rangle \cong \mathbb{Z}/6\mathbb{Z},$$

and the infinite three valent tree is generated by the action of $SL_2(\mathbb{Z})$ on the arc of the unit circle connecting i and $e^{i\pi/3}$. The arc of the unit circle connecting i and $e^{i\pi/3}$ (in fact every element of $\mathfrak{S}_1$) is stabilized by $-1 \in SL_2(\mathbb{Z})$.



The result that $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product of $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ is now a consequence of [Serre, Theorem 6] (see [Serre, 4.2(c)]).

THIS ACTION OF $SL_2(\mathbb{Z})$ on a 3-valent tree makes it very close to $G = SL_2(\mathbb{F}_3((t)))$?????, BECAUSE G IS THE AUTMORPHISM GROUP OF THE BUILDING G/I, WHERE I IS AN IWAHORI SUBGROUP AND THE BUILDING IS A 3 regular graph? FIGURE THIS OUT.

The group $SL_2(\mathbb{Z})$ is important in number theory because it is the $\ell = 1$ case of an abelian variety $\mathbb{C}^\ell/\Lambda$ which is a generalization of $\mathfrak{S}_1$, the upper half plane, and the automorphism group of the lattice is $Sp_2(\mathbb{Z}) \cong SL_2(\mathbb{Z})$.

Remark 16.9. The braid group on 3-strands $\mathcal{A}_3$ is presented by

$$\text{generators } \sigma_1, \sigma_2 \qquad \text{with relations} \quad \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

The center of $\mathcal{A}_3$ is cyclic generated by $(\sigma_1\sigma_2\sigma_1)^2$,

$$Z(\mathcal{A}_3) = \langle (\sigma_1\sigma_2\sigma_1)^2 \rangle \cong \mathbb{Z} \qquad \text{and} \qquad \frac{\mathcal{A}_3}{Z(\mathcal{A}_3)} \cong PSL_2(\mathbb{Z}).$$

DOES THIS MEAN THAT $SL_2(\mathbb{Z})$ and $\mathcal{A}_3$ ARE BOTH (different) CENTRAL EXTENSIONS OF $PSL_2(\mathbb{Z})$ (with different cocycles? and different kernels?) FIGURE THIS OUT.

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