15 Crystals and Schur functions

15.1 The category of crystals

Fix $n \in \mathbb{Z}_{>0}$. For $i \in \{1, \ldots, n\}$ let $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith spot. Let

$$B(\Box) = \big(1 \stackrel{\tilde f_1}{\to} 2 \stackrel{\tilde f_2}{\to} 3 \stackrel{\tilde f_3}{\to} \cdots \stackrel{\tilde f_{n-1}}{\to} n\big) \qquad \text{with} \quad \mathrm{wt}(i) = \varepsilon_i.$$

A *crystal* is an element B of the category generated by $B(\Box)$ under direct sums and tensor products. Let us specify the data (objects and morphisms) in the category of crystals. A crystal is a (finite) set B with functions

$$\mathrm{wt}\colon B \to \mathbb{Z}^n \qquad \text{and} \qquad \tilde f_i\colon B \to B \cup \{0\}, \quad \text{for } i \in \{1, \ldots, n-1\}.$$

A *crystal morphism* from B_1 to B_2 is a function $\Phi\colon B_1 \to B_2$ such that

$$\mathrm{wt}(\Phi(b)) = \mathrm{wt}(b) \qquad \text{and} \qquad \tilde f_i(\Phi(b)) = \Phi(\tilde f_i b),$$

for $b \in B_1$ and $i \in \{1, \ldots, n-1\}$. An *isomorphism* of crystals is a morphism of crystals $\Phi\colon B_1 \to B_2$ such that

$$\text{the inverse function } \Phi^{-1}\colon B_2 \to B_1 \text{ exists and } \Phi^{-1} \text{ is a morphism of crystals.}$$

The *direct sum* of crystals B_1 and B_2 is

$$B_1 \oplus B_2 = B_1 \sqcup B_2 \qquad \text{with wt and } \tilde f_i \text{ inherited from } B_1 \text{ and } B_2.$$

The *crystal graph* of B is the labeled graph with

$$\text{Vertices: } B \qquad \text{and} \qquad \text{Labeled edges: } \quad b \stackrel{\tilde f_i}{\to} \tilde f_i b.$$

For $i \in \{1, \ldots, n-1\}$ define

$$\tilde e_i\colon B \to B \cup \{0\} \qquad \text{by} \quad \tilde e_i(\tilde f_i b) = b \ \text{ if } \tilde f_i b \neq 0,$$

and $\tilde e_i b = 0$ if there does not exist $b' \in B$ such that $b = \tilde f_i b'$. Let $b \in B$ and $i \in \{1, \ldots, n-1\}$. Define $d_i^+(b)$ and $d_i^-(b)$ by

$$\begin{array}{lcl} \tilde e_i^{d_i^+(b)} b \neq 0 & \text{and} & \tilde e_i^{d_i^+(b)+1} b = 0, \\ \tilde f_i^{d_i^-(b)} b \neq 0 & \text{and} & \tilde f_i^{d_i^-(b)+1} b = 0. \end{array}$$

Then

$$S_i(b) = (\tilde e_i^{d_i^+(b)} b \stackrel{\tilde f_i}{\to} \cdots \stackrel{\tilde f_i}{\to} \tilde e_i b \stackrel{\tilde f_i}{\to} b \stackrel{\tilde f_i}{\to} \tilde f_i b \stackrel{\tilde f_i}{\to} \cdots \stackrel{\tilde f_i}{\to} \tilde f_i^{d_i^-(b)} b) \qquad \text{is the } i\text{-string of } b. \qquad \text{(istring)}$$

The *tensor product* of crystals B_1 and B_2 is

$$B_1 \otimes B_2 = B_1 \times B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$$

with

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$$

and

$$\tilde f_i(b_1 \otimes b_2) = \begin{cases} \tilde f_i b_1 \otimes b_2, & \text{if } d_i^+(b_1) > d_i^-(b_2), \\ b_1 \otimes \tilde f_i b_2, & \text{if } d_i^+(b_1) \le d_i^-(b_2), \end{cases} \qquad \text{(tensprodrule)}$$

15.1.1 Irreducible crystals and highest weight elements

Let B be a crystal.

- A *subcrystal* of B is a subset of B closed under the operators $\tilde{e}_i$ and $\tilde{f}_i$ (for $i \in \{1, \ldots, n-1\}$).
- A crystal is *irreducible*, or *simple*, if B has no subcrystals except $\emptyset$ and B.

A *highest weight element* in a crystal B is $b \in B$ such that

$$\text{if } i \in \{1, \ldots, n-1\} \quad \text{then} \quad \tilde{e}_i b = 0.$$

Let

$$B^+ = \{\text{highest weight elements of } B\} \qquad \text{and let} \qquad B^+_\lambda = \{b \in B^+ \mid \text{wt}(b) = \lambda\}, \quad \text{for } \lambda \in \mathbb{Z}^n.$$

Theorem 15.1.

(a) A crystal B is irreducible if and only if the crystal graph of B is connected.

(b) A crystal B is irreducible if and only if $\text{Card}(B^+) = 1$.

Proposition 15.2. *Assume B_1 and B_2 are irreducible crystals.*

(a) If $\Phi\colon B_1 \to B_2$ is a crystal morphism then Φ is a crystal isomorphism and $B_1 \cong B_2$.

(b) If $B_1^+ = \{b_1^+\}$ and $B_2^+ = \{b_2^+\}$ then

$$B_1 \cong B_2 \quad \textit{if and only if} \quad \text{wt}(b_1^+) = \text{wt}(b_2^+).$$

15.1.2 Characters

The *character* of a crystal B is

$$\text{char}(B) = \sum_{p \in B} x^{\text{wt}(p)}, \qquad \text{where} \quad x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$$

if $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$.

Theorem 15.3. *Let B_1 and B_2 be crystals.*

$$\text{char}(B_1 \oplus B_2) = \text{char}(B_1) + \text{char}(B_2), \qquad\qquad \text{char}(B_1 \otimes B_2) = \text{char}(B_1)\text{char}(B_2)$$

and

$$B_1 \cong B_2 \quad \textit{if and only if} \quad \text{char}(B_1) = \text{char}(B_2).$$

Proof. Decompose B_1 and B_2 into connected components. Let $B(\lambda)$ be the irreducible crystal of highest weight λ and let b_λ^+ denote the highest weight element of $B(\lambda)$ so that

$$B(\lambda)^+ = \{b_\lambda^+\} \qquad \text{and} \qquad \text{wt}(b_\lambda^+) = \lambda.$$

Then

$$B_1 \cong \bigsqcup_{p \in B_1^+} B(\text{wt}(p)) \cong B_2.$$

So

$$\text{char}(B_1) = \sum_{p \in B_1^+} \text{char}(B(\text{wt}(p))) = \text{char}(B_2).$$

□

15.1.3 Exercises

1. Show that if B_1 and B_2 are crystals then the action of $\tilde{e}_1, \ldots, \tilde{e}_{n-1}$ on $B_1 \otimes B_2$ is given by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } d_i^+(b_1) \geq d_i^-(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } d_i^+(b_1) < d_i^-(b_2), \end{cases}$$

2. Show that if B_1, B_2, B_3 are crystals then

$$\begin{array}{rccc} \Phi\colon & (B_1 \otimes B_2) \otimes B_3 & \longrightarrow & B_1 \otimes (B_2 \otimes B_3) \\ & b_1 \otimes b_2 \otimes b_3 & \longmapsto & b_1 \otimes b_2 \otimes b_3 \end{array} \qquad \text{is a crystal isomorphism.}$$

15.2 The crystal of words $B(\square)^{\otimes k}$

For $i \in \{1, \ldots, n\}$ let $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith spot. Let

$$B(\square) = \left(1 \xrightarrow{\tilde{f}_1} 2 \xrightarrow{\tilde{f}_2} 3 \xrightarrow{\tilde{f}_3} \cdots \xrightarrow{\tilde{f}_{n-1}} n\right) \qquad \text{with} \quad \mathrm{wt}(i) = \varepsilon_i.$$

A *word of length k in the alphabet* $\{b_1, \ldots, b_n\}$ is an element of

$$B(\square)^{\otimes k} = \{b_{i_1} \otimes \cdots \otimes b_{i_k} \mid i_1, \ldots, i_k \in \{1, \ldots, n\}\}.$$

The following theorem gives a direct description of the crystal $B(\square)^{\otimes k}$ as determined by the tensor product rule in (tensprodrule). By the second identity in Theorem 15.3,

$$\mathrm{char}(B(\square)^{\otimes k}) = (x_1 + \cdots + x_n)^k.$$

Theorem 15.4. *Let $b_{i_1} \otimes \cdots \otimes b_{i_k} \in B(\square)^{\otimes k}$ and fix $j \in \{1, \ldots, n-1\}$. Convert the word to a path by setting*

$$b_j = \nearrow, \quad b_{j+1} = \searrow \quad \textit{and} \quad b_r = \rightarrow \quad \textit{for } r \notin \{j, j+1\}.$$

Let ℓ be the position of the last $\nearrow$ step from the minimum height (marked in red in the example below). Then

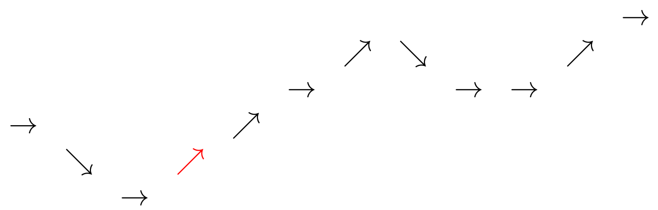
$$\begin{array}{c} \tilde{f}_j(b_{i_1} \otimes \cdots \otimes b_{i_k}) \textit{ is the same as } b_{i_1} \otimes \cdots \otimes b_{i_k} \\ \textit{except with } b_{i_\ell} \textit{ changed from } b_j \textit{ to } b_{j+1}, \end{array} \qquad \text{(fiBotk)}$$

and

$$\mathrm{wt}(b_{i_1} \otimes \cdots \otimes b_{i_k}) = \varepsilon_{i_1} + \cdots + \varepsilon_{i_k} = \mu_1 \varepsilon_1 + \cdots + \mu_n \varepsilon_n,$$
$$\textit{where} \quad \mu_i = (\# \textit{ of } b_j \textit{ in the word } b_{i_1} \otimes \cdots \otimes b_{i_k}).$$

For example, if $i = 2$ then

$$b_1 \quad b_3 \quad b_4 \quad b_2 \quad b_2 \quad b_1 \quad b_2 \quad b_3 \quad b_4 \quad b_1 \quad b_2 \quad b_4$$



Then

$$\begin{aligned} &\tilde{f}_2(b_1 \otimes b_3 \otimes b_4 \otimes b_2 \otimes b_2 \otimes b_1 \otimes b_2 \otimes b_3 \otimes b_4 \otimes b_1 \otimes b_2 \otimes b_4) \\ &= b_1 \otimes b_3 \otimes b_4 \otimes b_3 \otimes b_2 \otimes b_1 \otimes b_2 \otimes b_3 \otimes b_4 \otimes b_1 \otimes b_2 \otimes b_4. \end{aligned}$$

15.2.1 Highest weight elements of $B(\square)^{\otimes k}$

A word $p = b_{i_1} \otimes \cdots \otimes b_{i_k}$ is a *lattice permutation* if p satisfies

$$\text{if } r \in \{1, \ldots, k\} \text{ and } i \in \{1, \ldots, n-1\} \text{ then } \quad \#(b_i \text{ in } b_{i_1} \otimes \cdots \otimes b_{i_r}) \geq \#(b_{i+1} \text{ in } b_{i_1} \otimes \cdots \otimes b_{i_r}).$$

The following proposition gives three characterizations of the highest weight elements of $B(\square)^{\otimes k}$.

Theorem 15.5. *Let $p = b_{i_1} \otimes \cdots \otimes b_{i_k} \in B(\square)^{\otimes k}$ so that p is a word of length k.*

(a) Identify p with the path of length k in $\mathbb{Z}^n$ given by

$$0 \to \varepsilon_{i_1} \to (\varepsilon_{i_1} + \varepsilon_{i_2}) \to \cdots \to (\varepsilon_{i_1} + \cdots + \varepsilon_{i_k}).$$

The word p is a highest weight element in $B(\square)^{\otimes k}$ if and only if the path is completely in the region

$$C - \rho = \{(\mu_1, \ldots, \mu_n) \in (\mathbb{Z}_{\geq 0})^n \mid \text{if } i \in \{1, \ldots, n-1\} \text{ then } \mu_i - \mu_{i+1} > -1\}.$$

(b) The word p is a lattice permutation if and only if p is a highest weight element in $B(\square)^{\otimes k}$.

(c) Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{Z}_{\geq 0})^n$ with $\lambda_1 + \cdots + \lambda_n = k$ and let $\big(B(\square)^{\otimes k}\big)^+_\lambda$ be the set of highest weight elements of $B(\square)^{\otimes k}$ which are of weight λ.

(ca) If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then

$$\begin{matrix} \left\{ \begin{matrix} \text{standard tableaux} \\ \text{of shape } \lambda \end{matrix} \right\} & \longleftrightarrow & \big(B(\square)^{\otimes k}\big)^+_\lambda \\ Q & \longmapsto & b_{r(Q^{-1}(1))} \otimes \cdots \otimes b_{r(Q^{-1}(k))} \end{matrix} \qquad \text{is a bijection.}$$

where $r(Q^{-1}(i))$ denotes the row index of the entry i in the standard tableau Q.

(cb) If λ does not satisfy $\lambda_1 \geq \cdots \geq \lambda_n$ then $\big(B(\square)^{\otimes k}\big)^+_\lambda = \emptyset$.

15.2.2 Exercises

1. Carefully prove that the action of $\tilde{f}_i$ on $B(\square)^{\otimes k}$ given in (fiBotk) follows from the definition of the tensor product in (tensprodrule).
2. Give an analogue of (fiBotk) which describes the action of $\tilde{e}_i$ on $B(\square)^{\otimes k}$.
3. Give a careful proof of Theorem 15.5.
4. Illustrate Theorem 15.5(c) by expliciitly checking that the map sends the 5 standard tableaux of shape $\lambda = (221)$ to highest weight vectors in $B(\square)^{\otimes 5}$.
5. Let C be the region in $(\mathbb{Z}_{\geq 0})^n$ defined in Proposition 15.5. Let

   $$\omega_1 = \varepsilon_1, \quad \omega_2 = \varepsilon_1 + \varepsilon_2, \quad \ldots, \quad \omega_n = \varepsilon_1 + \cdots + \varepsilon_n.$$

   Show that $C = \mathbb{Z}_{\geq 0}\text{-span}\{\omega_1, \ldots, \omega_n\}$.

15.3 The irreducible crystals $B(\lambda)$

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. A *semistandard Young tableau, or SSYT, of shape λ filled from* $\{1, \ldots, n\}$ is a function $T\colon \lambda \to \{1, \ldots, n\}$ such that

(a) If $(r, c), (r, c+1) \in \lambda$ then $T(r, c) \leq T(r, c+1)$,

(b) If $(r, c), (r+1, c) \in \lambda$ then $T(r, c) < T(r+1, c)$.

The *arabic reading* of a SSYT T is the word formed by reading the entries of T from right to left in rows, r ow by row from top to bottom. For example the SSYT

1	1	1	2	3
2	2	4	4	
3	4			

has arabic reading $\quad b_3 \otimes b_2 \otimes b_1 \otimes b_1 \otimes b_1 \otimes b_4 \otimes b_4 \otimes b_2 \otimes b_2 \otimes b_4 \otimes b_3$

To control the spacing and formatting it is often helpful to drop the $\otimes$ symbols and write

$$b_3 \otimes b_2 \otimes b_1 \otimes b_1 \otimes b_1 \otimes b_4 \otimes b_4 \otimes b_2 \otimes b_2 \otimes b_4 \otimes b_3 = b_3 b_2 b_1 b_1 b_1 b_4 b_4 b_2 b_2 b_4 b_3.$$

Theorem 15.6. *Let $B(\lambda) = \{SSYTs\ of\ shape\ \lambda\ filled\ from\ \{1, \ldots, n\}\}$.*

(a) There is a unique crystal structure on $B(\lambda)$ such that

$$\begin{array}{ccc} B(\lambda) & \longrightarrow & B(\Box)^{\otimes k} \\ p & \longmapsto & (\textit{arabic reading of } p) \end{array} \qquad \textit{is a crystal morphism.}$$

(b) $B(\lambda)$ is an irreducible crystal of highest weight λ.

15.4 $B(\lambda/\mu)$ and the Littlewood-Richardson rule

Let λ and μ be partitions such that $\mu \subseteq \lambda$, where λ and μ are viewed as sets of boxes. Let λ/μ be the complement of μ in λ (as sets of boxes) and let k be the number of boxes in λ/μ. Let

$$B(\lambda/\mu) = \{\text{SSYT of shape } \lambda/\mu \text{ filled from } \{1, \ldots, n\}\}.$$

There is a unique crystal structure on $B(\lambda/\mu)$ such that

$$\begin{array}{ccc} B(\lambda/\mu) & \longrightarrow & B(\Box)^{\otimes k} \\ p & \longmapsto & (\text{arabic reading of } p) \end{array} \qquad \text{is a crystal morphism.} \qquad \text{(Bskewshape)}$$

Let $\nu = (\nu_1, \ldots, \nu_n) \in (\mathbb{Z}_{\geq 0})^n$. A *Littlewood-Richardson filling of λ/μ with content ν* is an element $p \in B(\lambda/\mu)_\nu$ such that the arabic reading of p is a lattice permutation. The following result is a corollary of Theorem 15.5.

Theorem 15.7. *(Littlewood-Richardson rule) Using (Bskewshape) to identify elements of $B(\lambda/\mu)$ with words,*

$$B(\lambda/\mu)^+_\nu = \{\textit{Littlewood-Richardson fillings of } \lambda/\mu \textit{ with content } \nu\}.$$

15.5 Plactic operators (crystal Schur-Weyl duality)

The *plactic operators* $\tilde{s}_1, \ldots, \tilde{s}_{k-1}$ are the partial functions $\tilde{s}_i \colon B(\Box)^{\otimes k} \to B(\Box)^{\otimes k}$ defined by

$$\tilde{s}_j(b_{i_1} \cdots b_{i_j} b_{i_{j+1}} \cdots b_{i_k}) = \begin{cases} (b_{i_1} \cdots b_{i_{j+1}} b_{i_j} \cdots b_{i_k}), & \text{if } i_j \leq i_{j+2} < i_{j+1} \text{ or } i_{j+1} < i_{j-1} \leq i_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $M = M_{n^k \times n^k}(\{0,1\})$ be the monoid of $n^k \times n^k$ matrices with entries in $\{0,1\}$. View $\tilde{s}_1, \ldots, \tilde{s}_k$ as elements of M and

$$\text{let } \tilde{S}_k \text{ be the submonoid of } M \text{ generated by } \tilde{s}_1, \ldots, \tilde{s}_{k-1}.$$

Similarly,

$$\text{let } \tilde{U} \text{ be the submonoid of } M \text{ generated by } \tilde{e}_1, \ldots, \tilde{e}_{n-1}, \tilde{f}_1, \ldots, \tilde{f}_{n-1}.$$

Let

$$\hat{S}_n^\lambda = \{\text{standard tableaux of shape } \lambda\} \qquad \text{and let} \quad \tilde{s}_1, \ldots, \tilde{s}_n \in \mathrm{End}(\hat{S}_n^\lambda)$$

be given by

$$\tilde{s}_i Q = (Q \text{ with the position of } i \text{ and } i+1 \text{ switched}), \qquad \text{if the result is a standard tableau,}$$

and $\tilde{s}_i Q = 0$, otherwise.

Theorem 15.8.

(a) If $b_{i_1} \cdots b_{i_k} \in B(\Box)^{\otimes k}$ and $i \in \{1, \ldots, k-1\}$ and $j \in \{1, \ldots, n-1\}$ then

$$\tilde{s}_i \tilde{f}_j (b_{i_1} \cdots b_{i_k}) = \tilde{f}_j \tilde{s}_i (b_{i_1} \cdots b_{i_k}).$$

(b) Under the $\tilde{S}_k \times \tilde{U}$ action

$$B(\Box)^{\otimes k} \cong \bigsqcup_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} \tilde{S}_n^\lambda \times B(\lambda).$$

15.5.1 Exercises

1. Let $e_1, \ldots, e_{n-1}$ and $f_1, \ldots, f_{n-1}$ be matrices in $M_{n^k \times n^k}(\{0,1\})$ which describe the action of the root operators on $B(\Box)^{\otimes k}$. Let M be the subsemigroup of $M_{n^k \times n^k}(\{0,1\})$ generated by $e_1, \ldots, e_{n-1}$ and $f_1, \ldots, f_{n-1}$. Find relations satisfied by $e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1}$ which provide a presentation of M.

2. **Knuth Equivalence.** (also called plactic equivalence) Define an equivalence relation on words as follows: If x, z, z are letters (elements of $B(\Box)$) then

   (a) If $x \leq y < z$ then $xzy \equiv zxy$,

   (b) If $x < y \leq z$ then $yzx \equiv yxz$.

   Show that Knuth equivalence describes the orbits of the action of the monoid $\tilde{S}_k$ on $B(\Box)^{\otimes k}$.

15.6 Schur functions and the Weyl character formula

For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ let

$$x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}, \qquad \text{so that } \{x^\mu \mid \mu \in \mathbb{Z}^n\} \text{ is a } \mathbb{C}\text{-basis of } \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

For $i \in \{1, \ldots, n-1\}$ define $s_i \colon \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ by

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \ldots, x_n).$$

The operators $s_1, \ldots, s_{n-1}$ define an action of S_n on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. A *symmetric function* is an element of

$$\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n} = \{f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid \text{if } w \in S_n \text{ then } wf = f\}.$$

Let $\varepsilon = \sum_{w \in S_n} \det(w) w$ and define

$$a_\mu = \varepsilon(x^\mu) = \sum_{w \in S_n} \det(w) w x^\mu = \sum_{w \in S_n} \det(w) x_{w(1)}^{\mu_1} \cdots x_{w(n)}^{\mu_n}.$$

Let $\rho = (n-1, n-2, \ldots, 2, 1, 0)$. The *Schur functions* are

$$s_\mu = \frac{a_{\mu+\rho}}{a_\rho}, \qquad \text{for } \mu \in \mathbb{Z}^n. \qquad \text{(schurdef)}$$

Theorem 15.9. *Let B be a crystal with weight function* $\mathrm{wt} \colon B \to \mathbb{Z}^n$.

(a) $\mathrm{char}(B)$ *is symmetric function.*

(b) Let s_μ be the Schur function for $\mu \in \mathbb{Z}^n$ and let B^+ be the set of highest weight elements of B. Then

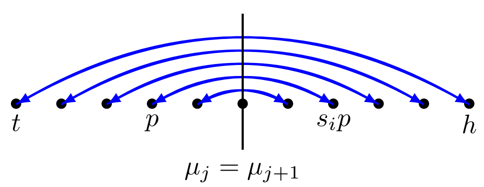
$$\mathrm{char}(B) = \sum_{p \in B^+} s_{\mathrm{wt}(p)}.$$

Proof. Let B be a crystal. Let $p \in B$ and $j \in \{1, \ldots, n-1\}$. Let $d_j^+(p)$, $d_j^-(p)$ and $S_j(p)$ be as defined in (istring) so that the *i-string* of p is

$$\{\tilde{f}_j^{d_j^-(p)} p, \ldots, \tilde{f}_j p, p, \tilde{e}_j p, \ldots, \tilde{e}_j^{d_j^+(p)} p\} = S_j(p).$$

Let $s_j p$ be the element of $S_j(p)$ such that

$$\mathrm{wt}(s_j p) = s_j \mathrm{wt}(p).$$



with $t = \tilde{f}_j^{d_i^-(p)} p$, $h = \tilde{e}_j^{d_i^+(p)} p$.

Then $s_j(s_j(p)) = p$ and

$$s_j \mathrm{char}(B) = \sum_{p \in B} x^{s_j \mathrm{wt}(p)} = \sum_{p \in B} x^{\mathrm{wt}(s_j p)} = \mathrm{char}(B).$$

So $\mathrm{char}(B)$ is a symmetric function.

Since $\text{char}(B)$ is a symmetric function then

$$\text{char}(B)a_\rho = \text{char}(B)\varepsilon(x^\rho) = \varepsilon(\text{char}(B)x^\rho)$$

and

$$\begin{aligned}\text{char}(B) &= \frac{1}{a_\rho}\text{char}(B)a_\rho = \frac{\varepsilon(\text{char}(B)x^\rho)}{a_\rho} \\ &= \sum_{p\in B}\frac{\varepsilon(x^{\text{wt}(p)+\rho})}{a_\rho} = \sum_{p\in B}\frac{a_{\text{wt}(p)+\rho}}{a_\rho} = \sum_{p\in B} s_{\text{wt}(p)}.\end{aligned} \qquad (*)$$

There is some cancellation which can occur in this sum.

Let $p = b_{i_1}\otimes\cdots\otimes b_{i_k}$ be a word in B. View p as the path in $(\mathbb{Z}_{\ge 0})^n$ given by

$$0 \to \varepsilon_1 \to (\varepsilon_{i_1}+\varepsilon_{i_2}) \to \cdots \to (\varepsilon_{i_1}+\cdots+\varepsilon_{i_k}).$$

Assume that this path is not completely contained in the cone

$$C-\rho = \{(\mu_1,\ldots,\mu_n)\in(\mathbb{Z}_{\ge 0})^n \mid \text{if } i\in\{1,\ldots,n-1\} \text{ then } \mu_i-\mu_{i+1} > -1\}.$$

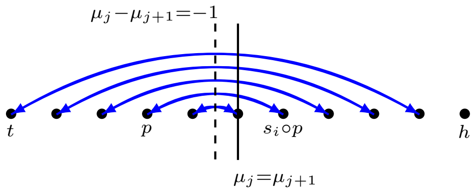
Let t be the first time that p leaves the cone $C-\rho$. In other words, let $t\in\{1,\ldots,k\}$ such that there exists $j\in\{1,\ldots,n-1\}$ such that

$$\mu = \varepsilon_{i_1}+\cdots+\varepsilon_{i_t} \quad \text{has} \quad \mu_j-\mu_{j+1} = -1.$$

Let $p(t) = b_{i_1}\otimes\cdots\otimes b_{i_t}$.

Let j be the minimal index such that $\mu_j-\mu_{j+1}=-1$ and define $s_j\circ p$ to be the element of the j-string of p such that

$$\text{wt}(s_j\circ p) = s_j\circ p.$$



Since $\mu_j-\mu_{j+1}=-1$ then $\tilde{e}_j p\neq 0$ and $s_j\circ p$ is well defined.

If $q = s_j\circ p = b_{i'_1}\otimes\cdots\otimes b_{i'_k}$ then the first time t that q leaves the cone $C-\rho$ is the same as the first time that p leaves the cone $C-\rho$ and

$$p(t) = \varepsilon_{i_1}+\cdots+\varepsilon_{i_t} = \varepsilon_{i'_1}+\cdots+\varepsilon_{i'_t} = q(t).$$

Thus $s_i\circ q = p$ and $s_i\circ(s_i\circ p) = p$.

Since

$$s_{\text{wt}(s_i\circ p)} = s_{s_i\circ\text{wt}(p)} = -s_{\text{wt}(p)}$$

the terms $s_{\text{wt}(s_i\circ p)}$ and $s_{\text{wt}(p)}$ cancel in the sum in (*). Thus

$$\text{char}(B) = \sum_{\substack{p\in B\\ p\subseteq C-\rho}} s_{\text{wt}(p)}.$$

□

Corollary 15.10. *(The Weyl character formula) Let λ be a partition and let $B(\lambda)$ be the irreducible crystal of highest weight λ. Then*

$$\text{char}(B(\lambda)) = s_\lambda.$$

Proof. The element b_λ^+ is the unique highest weight element of $B(\lambda)$. Thus, by Theorem 15.9, $\text{char}(B(\lambda)) = s_\lambda$. □

15.6.1 Exercises

1. Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ and let $w \in S_n$.

   (a) Show that
$$a_\mu = \det \begin{pmatrix} x_1^{\mu_1} & x_1^{\mu_2} & \cdots & x_1^{\mu_n} \\ x_2^{\mu_1} & x_2^{\mu_2} & \cdots & x_2^{\mu_n} \\ \vdots & \vdots & & \vdots \\ x_n^{\mu_1} & x_n^{\mu_2} & \cdots & x_n^{\mu_n} \end{pmatrix}$$

   (b) Show that
$$wa_\mu = (-1)^{\ell(w)} a_\mu.$$

   (c) Show that $a_\mu = 0$ if and only if there exists $i, j \in \{1, \ldots, n\}$ with $i < j$ such that $s_{ij}\mu = \mu$.

   (d) Let
$$(\mathbb{Z}^n)^+ = \{(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \mid \mu_1 \geq \cdots \geq \mu_n\},$$
$$(\mathbb{Z}^n)^{++} = \{(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \mid \mu_1 > \cdots > \mu_n\}.$$
   Let $\rho = (n-1, n-2, \ldots, 2, 1, 0)$. Show that
$$\begin{matrix} (\mathbb{Z}^n)^+ & \longrightarrow & (\mathbb{Z}^n)^{++} \\ \lambda & \longmapsto & \lambda + \rho \end{matrix} \qquad \text{is a bijection.}$$

   (e) Show that
$$a_\rho = \prod_{i<j} (x_j - x_i).$$

   (f) Show that a_μ is divisible by a_ρ (i.e. show that $\frac{a_\mu}{a_\rho} \in \mathbb{Z}[x_1, \ldots, x_n]$).

   (g) Show that
$$\begin{matrix} \mathbb{C}[x_1, \ldots, x_n]^{S_n} & \longrightarrow & \mathbb{C}[x_1, \ldots, x_n]^{\det} \\ f & \longmapsto & a_\rho f \end{matrix} \qquad \text{is a vector space isomorphism.}$$

   (h) Show that $\{a_\mu \mid \mu \in (\mathbb{Z}^n)^{++}\}$ is a $\mathbb{C}$-basis of $\mathbb{C}[x_1, \ldots, x_n]^{\det}$.

   (i) Show that $\{s_\lambda \mid \lambda \in (\mathbb{Z}^n)^+\}$ is a $\mathbb{C}$-basis of $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$.

2. Let $\mu \in \mathbb{Z}^n$. Define
$$w \circ \mu = w(\mu + \rho) - \rho, \qquad \text{for } w \in S_n.$$

   (a) Prove that
$$s_{w \circ \mu} = (-1)^{\ell(w)} s_\mu.$$

   (b) Prove that $s_\mu = 0$ if and only if there exists $i, j \in \{1, \ldots, n\}$ with $i \neq j$ such that $s_{ij} \circ \mu = \mu$.

   (c) Prove that $s_\mu = 0$ if and only if there exists $i, j \in \{1, \ldots, n\}$ with $i \neq j$ such that $\mu_i = \mu_j - 1$.

15.7 Examples

15.7.1 Row and column crystals

Example 15.1.

$$B(1^k) = \left\{ \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array} \;\middle|\; \begin{array}{c} i_1, \ldots, i_k \in \{1, \ldots, n\} \\ i_1 < \cdots < i_k \end{array} \right\}$$

with

$$\tilde{f}_j\left(\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array} \right) = \begin{cases} \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline j \\ \hline \vdots \\ \hline i_k \\ \hline \end{array}, & \text{if } j \in \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array}, \\ 0, & \text{if } j \notin \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array}. \end{cases}$$

and

$$\mathrm{char}(B(1^k)) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} = e_k.$$

Example 15.2.

$$B((k)) = \left\{ \begin{array}{|c|c|c|}\hline i_1 & \cdots & i_k \\ \hline \end{array} \;\middle|\; \begin{array}{c} i_1, \ldots, i_k \in \{1, \ldots, n\} \\ i_1 \le \cdots \le i_k \end{array} \right\}$$

with

$$\tilde{f}_j\left(\begin{array}{|c|c|c|}\hline i_1 & \cdots & i_k \\ \hline \end{array} \right) = \begin{cases} \begin{array}{|c|c|c|c|c|}\hline i_1 & \cdots & \underbrace{j \cdots j}_{r-1} & \cdots & i_k \\ \hline \end{array}, & \text{if } j \text{ appears } r \text{ times in } \begin{array}{|c|c|c|}\hline i_1 & \cdots & i_k \\ \hline \end{array}, \\ 0, & \text{if } j \text{ does not appear in } \begin{array}{|c|c|c|}\hline i_1 & \cdots & i_k \\ \hline \end{array}, \end{cases}$$

and

$$\mathrm{char}(B((k)) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \cdots x_{i_k} = h_k.$$

15.7.2 Word crystals for $n = 3$ and $k \in \{1, 2, 3\}$

Let $B = B(\square)$.

$$B = B^{\otimes 1} \qquad\qquad 1 \xrightarrow{\tilde{f}_1} 2 \xrightarrow{\tilde{f}_2} 3$$

$$B \otimes B = B^{\otimes 2} \qquad \begin{array}{ccccc} 11 & \xrightarrow{1} & 21 & \xrightarrow{1} & 22 \\ & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\ 12 & & 31 & \xrightarrow{1} & 32 \\ \downarrow{\scriptstyle 2} & & & & \downarrow{\scriptstyle 2} \\ 13 & \xrightarrow{1} & 23 & & 33 \end{array}$$

$B \otimes B \otimes B = B^{\otimes 3}$

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$$\begin{array}{ccccccc} 111 & \xrightarrow{1} & 211 & \xrightarrow{1} & 221 & \xrightarrow{1} & 222 \\ & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\ & & 311 & \xrightarrow{1} & 321 & \xrightarrow{1} & 322 \\ & & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\ & & & & 331 & \xrightarrow{1} & 332 \\ & & & & & & \downarrow{\scriptstyle 2} \\ & & & & & & 333 \end{array}$$

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2

$$\begin{array}{ccccccc} 121 & \xrightarrow{1} & 122 & \xrightarrow{2} & 132 & \xrightarrow{2} & 133 \\ \downarrow{\scriptstyle 2} & & & & & & \downarrow{\scriptstyle 1} \\ 131 & \xrightarrow{1} & 231 & \xrightarrow{1} & 232 & \xrightarrow{2} & 233 \end{array}$$

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3

$$\begin{array}{ccccccc} 112 & \xrightarrow{1} & 212 & \xrightarrow{2} & 312 & \xrightarrow{2} & 313 \\ \downarrow{\scriptstyle 2} & & & & & & \downarrow{\scriptstyle 1} \\ 113 & \xrightarrow{1} & 213 & \xrightarrow{2} & 223 & \xrightarrow{2} & 323 \end{array}$$

1
2
3

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