13 Definitions of the symmetric functions

13.1 The power sum symmetric functions p_μ

Eefine p_r for $r \in \mathbb{Z}_{\geq 0}$ by

$$p_r = x_1^r + x_2^r + \cdots + x_n^r \qquad \text{and define} \qquad p_\nu = p_{\nu_1} p_{\nu_2} \cdots p_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers.

13.2 The elementary symmetric functions e_μ

Define e_r for $r \in \mathbb{Z}_{\geq 0}$ by

$$\sum_{r \in \mathbb{Z}_{\geq 0}} e_r z^r = \prod_{i=1}^n (1 + x_i z) \qquad \text{and define} \qquad e_\nu = e_{\nu_1} e_{\nu_2} \cdots e_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers.

13.3 The homogeneous symmetric functions h_μ

Define h_r for $r \in \mathbb{Z}_{\geq 0}$ by

$$\sum_{r \in \mathbb{Z}_{\geq 0}} h_r z^r = \prod_{i=1}^n \frac{1}{1 - x_i z} \qquad \text{and define} \qquad h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers.

13.4 The little q's

Following [Mac, (Ch. III (2.10)], define q_r for $r \in \mathbb{Z}_{\geq 0}$ by

$$\sum_{r \in \mathbb{Z}_{\geq 0}} q_r z^r = \prod_{i=1}^n \frac{1 - t x_i z}{1 - x_i z} \qquad \text{and define} \qquad q_\nu = q_{\nu_1} q_{\nu_2} \cdots q_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers. In plethystic notation CHECK THIS

$$q_\nu = e_\nu[X(t-1)] \qquad \text{and} \qquad e_\nu = q_\nu\Big[\frac{X}{1-t}\Big].$$

13.5 The little g's

For a symbol a define the infinite product

$$(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \cdots .$$

Define g_r for $r \in \mathbb{Z}_{\geq 0}$ by

$$\sum_{r \in \mathbb{Z}_{\geq 0}} g_r z^r = \prod_{i=1}^n \frac{(t x_i z; q)_\infty}{(x_i z; q)_\infty} \qquad \text{and define} \qquad g_\nu = g_{\nu_1} g_{\nu_2} \cdots g_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers.

13.6 The nonsymmetric Macdonald polynomials E_μ

Let $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The symmetric group S_n acts on $\mathbb{C}[X]$ by permuting $x_1, \ldots, x_n$. Let

$$\mathbb{C}[X]^{S_n} = \{g \in \mathbb{C}[X] \mid \text{if } w \in S_n \text{ then } wg = g\} \qquad \text{the ring of symmetric functions.}$$

Letting $s_1, \ldots, s_{n-1}$ denote the simple transpositions in S_n,

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n).$$

For $f \in \mathbb{C}[X]$ and $i \in \{1, \ldots, n-1\}$ define

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}.$$

Let $\mathbb{Z}^n_{\geq 0}$ denote the set of length n sequences $\mu = (\mu_1, \ldots, \mu_n)$ of nonnegative integers (sometimes called the set of weak compositions). Define E_μ for $\mu \in \mathbb{Z}^n_{\geq 0}$ by setting $E_{(0,0,\ldots,0)} = 1$ and using the following recursions:

(1) If $\mu_i > \mu_{i+1}$ then $\qquad E_{s_i\mu} = \Big(\partial_i x_i - t x_i \partial_i + \dfrac{(1-t) q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}\Big) E_\mu,$
where $v_\mu \in S_n$ is minimal length such that $v_\mu \mu$ is weakly increasing, and

(2) $E_{(\mu_n+1, \mu_1, \ldots, \mu_{n-1})} = q^{\mu_n} x_n E_\mu(x_2, \ldots, x_n, q^{-1} x_1)$.

Explicitly, the permutation $v_\mu \in S_n$ which is minimal length such that $v_\mu \mu$ is weakly increasing is given by

$$v_\mu(i) = 1 + \#\{i' \in \{1, \ldots, i-1\} \mid \mu_{i'} \leq \mu_i\} + \#\{i' \in \{i+1, \ldots n\} \mid \mu_{i'} < \mu_i\}.$$

13.7 The symmetric Macdonald polynomials P_λ

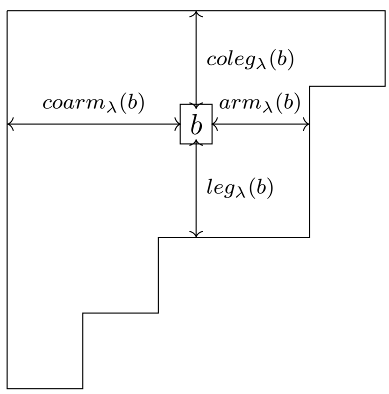
Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Define

$$P_\lambda(q,t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w\Big(E_\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}\Big),$$

where $W_\lambda(t)$ is the appropriate constant which makes the coefficient of x^λ equal to 1 in $P_\lambda(q,t)$.

13.8 The big Js and the big Qs

Let λ be a partition and let λ' denote the conjugate partition to λ. Following, [Mac, VI (6.14)] for a box $b = (i,j)$ in λ define



$$\mathrm{coleg}_\lambda(b) = i - 1,$$

$$\mathrm{coarm}_\lambda(b) = j - 1, \qquad b = (i,j), \qquad \mathrm{arm}_\lambda(b) = \lambda_i - j,$$

$$\mathrm{leg}_\lambda(b) = \lambda'_j - i.$$

The *hook length* $h(b)$ and the *content* $c(b)$ of the box b are defined by

$$h(b) = \mathrm{arm}_\lambda(b) + \mathrm{leg}_\lambda(b) + 1 \qquad \text{and} \qquad c(b) = \mathrm{coarm}_\lambda(b) - \mathrm{coleg}_\lambda(b).$$

Define the *upper and lower hooks of a box* and the *upper and lower hook products of a partition* by

$$h^*_\lambda(b) = 1 - q^{\mathrm{arm}_\lambda(b)+1}t^{\mathrm{leg}_\lambda(b)}, \qquad h^\lambda_*(b) = 1 - q^{\mathrm{arm}_\lambda(b)}t^{\mathrm{leg}_\lambda(b)+1},$$

$$h^*_\lambda = \prod_{b\in\lambda} h^*_\lambda(b), \qquad h^\lambda_* = \prod_{b\in\lambda} h^\lambda_*(b),$$

The *integral form Macdonald polynomials* J_μ and the *dual Macdonald polynomials* Q_μ are given by [Mac, (8.3) and (8.11)]:

$$J_\mu(q,t) = h^\mu_* P_\mu(q,t) \qquad \text{and} \qquad Q_\mu(q,t) = \frac{h^\mu_*}{h^*_\mu} P_\mu(q,t).$$

13.9 The fermionic Macdonald polynomials $A_{\lambda+\delta}$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_n$ define

$$\lambda + \delta = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_{n-1} + 1, \lambda_n)$$

and

$$A_{\lambda+\delta}(q,t) = \Big(\prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\Big) \sum_{w\in S_n} (-1)^{\ell(w)} wE_{\lambda+\delta}.$$

Theorem 13.1. *(Weyl character formula for Macdonald polynomials)*

$$A_\delta(q,t) = \prod_{i<j}(x_i - tx_j) \qquad \textit{and} \qquad P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}.$$

13.10 The Schurs s_λ and the Big Schurs S_λ

The *Schur functions* s_λ and the *Big Schurs* S_λ are given in [Mac, Ch. I (7.7) and Ch. VI (8.9)] by the formulas

$$s_\lambda = \sum_\rho \frac{1}{z_\rho} \chi^\lambda_{S_n}(\rho) p_\rho \qquad \text{and} \qquad S_\lambda = S_\lambda(x;t) = \sum_\rho \frac{1}{z_\rho}\chi^\lambda_{S_n}(\rho)\Big(\prod_{i=1}^{\ell(\rho)}(1 - t^{\rho_i})\Big)p_\rho$$

where p_ρ is the power sum symmetric function and $\chi^\lambda_{S_n}$ are the irreducible characters of the symmetric group. In plethystic notation

$$S_\lambda = s_\lambda[X(1-t)] \qquad \text{and} \qquad s_\lambda = S_\lambda\Big[\frac{X}{1-t}\Big].$$

13.11 The modified Macdonald polynomials $\tilde{H}_\lambda(x;,q,t)$

Define $K_{\lambda\mu}(q,t)$ and the *modified Macdonald polynomials* $\widetilde{H}_\mu$ by the formulas

$$J_\mu = \sum_\mu K_{\lambda\mu}(q,t) S_\lambda \qquad \text{and} \qquad \widetilde{H}_\mu = \sum_\mu t^{n(\mu)} K_{\lambda\mu}(q,t^{-1}) s_\lambda. \qquad \text{(modMacdefn)}$$

In other words, change J_μ to $\widetilde{H}_\mu$ by changing S_λ to s_λ, changing t to t^{-1} and mutliplying by an overall factor of $t^{n(\mu)}$. This buries all the plethystic substitution into the switch from S_λ to s_λ. Write

$$\widetilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,t^{-1}) \qquad \text{so that} \qquad \widetilde{H}_\lambda(q,t;X) = \sum_\mu \widetilde{K}_{\lambda\mu}(q,t)s_\mu$$

The relation (modMacdefn) is not disssimilar to the relation

$$q_\mu = \sum_\lambda K_{\lambda\mu}S_\lambda. \qquad \text{and} \qquad h_\mu = \sum_\lambda K_{\lambda\mu}s_\lambda, \qquad \text{where} \quad K_{\lambda\mu} = K_{\lambda\mu}(0,1).$$

Remark 13.2. François Bergeron might define the *modified Macdonald polynomials* $\widetilde{H}_\mu = \widetilde{H}_\mu(q,t;x)$, using plethystic notation, by

$$\widetilde{H}_\mu(q,t;X) = t^{\eta(\mu)}P_\mu(\frac{X}{1-t^{-1}};q,t)\prod_{c\in\mu}(q^{a(c)} - t^{-(l(c)+1)}). \qquad \square$$

13.12 Transition matrices $\chi(t)$, $K(q,t)$, $Z(q,t)$, $\Psi(q,t)$ and $\mathcal{K}(q,t)$

Define $\chi_{\lambda\nu}(t)$ by

$$S_\lambda = \sum_\nu \chi_{\lambda\nu}(t)m_\nu.$$

Since $\chi_{\lambda\nu} = \langle S_\lambda(t), q_\nu(t)\rangle_{0,t}$ and $\langle q_\nu(t), m_\mu\rangle_{0,t} = \delta_{\nu\mu}$ and $\langle S_\lambda(t), s_\mu\rangle_{0,t} = \delta_{\lambda\mu}$ then

$$q_\nu(t) = \sum_\lambda \chi_{\lambda\nu}(t)s_\lambda,$$

Define $K_{\lambda\nu}(q,t)$ and $Z_{\lambda\mu}(q,t)$ by

$$J_\mu(q,t) = \sum_\lambda K_{\lambda\mu}(q,t)S_\lambda(t) \qquad \text{and} \qquad J_\lambda(q,t) = \sum_\mu Z_{\lambda\mu}(q,t)s_\mu.$$

Define $\Psi_{\mu\nu}(q,t)$ and $\mathcal{K}_{\lambda\mu}(q,t)$ by

$$J_\mu(q,t) = \sum_\mu \Psi_{\mu\nu}(q,t)m_\nu \qquad \text{and} \qquad J_\mu(q,qt) = \sum_\lambda \mathcal{K}_{\lambda\mu}(q,t)J_\lambda(q,t).$$

Remark 13.3. Relations: $\Psi(q,t) = Z(q,t)K(0,1)$ and $\Psi(q,t) = K(q,t)^t\chi(t)$. Since

$$s_\lambda = \sum_\mu K_{\lambda\mu}(0,1)m_\mu$$

then

$$\Psi_{\lambda\nu}(q,t) = \sum_\mu Z_{\lambda\mu}(q,t)K_{\mu\nu}(0,1), \qquad \text{and} \qquad \Psi_{\mu\nu}(q,t) = \sum_\lambda K_{\lambda\mu}(q,t)\chi_{\lambda\nu}(t). \qquad \square$$

Remark 13.4. A difference equation: $D_t\Psi = \mathcal{K}\Psi$ so that $\mathcal{K}$ is a connection matrix! Since

$$D_t\Psi = \Psi(q,qt) = \mathcal{K}(q,t)\Psi(q,t) = \mathcal{K}\Psi \qquad \text{and} \qquad D_tZ = Z(q,qt) = \mathcal{K}(q,t)Z(q,t) = \mathcal{K}Z,$$

then Ψ and Z a both solutions of the same difference equation, but with different initial conditions,

$$\Psi(q,q) = K(0,1) \qquad \text{and} \qquad Z(q,q) = \text{id}. \qquad \square$$

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