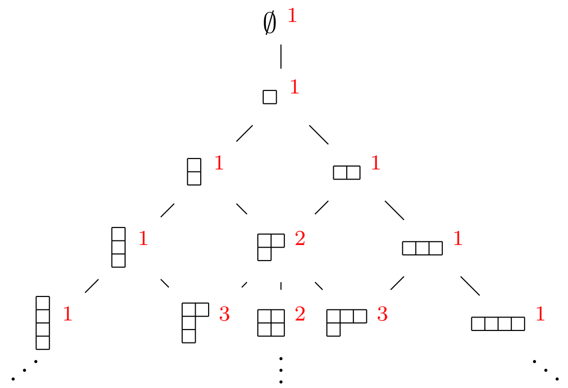
21 Partitions

21.1 Partitions and the Young lattice

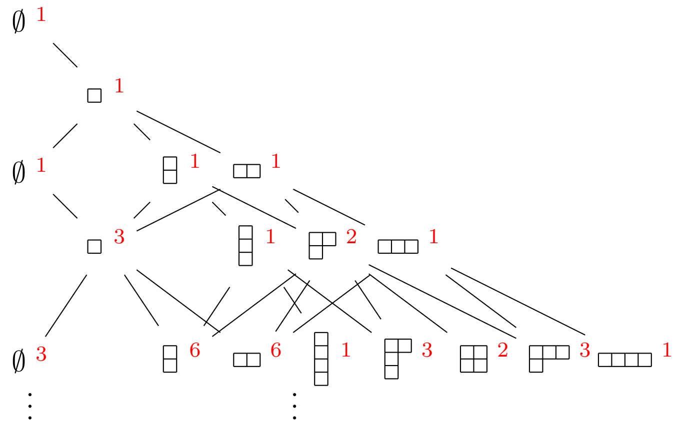
Young's lattice (boxes in a corner)



$$1^2 = 1 = 0!$$
$$1^2 = 1 = 1!$$
$$1^2 + 1^2 = 2 \cdot 1 = 2!$$
$$1^2 + 2^2 + 1^2 = 3 \cdot 2 \cdot 1 = 3!$$
$$1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$$

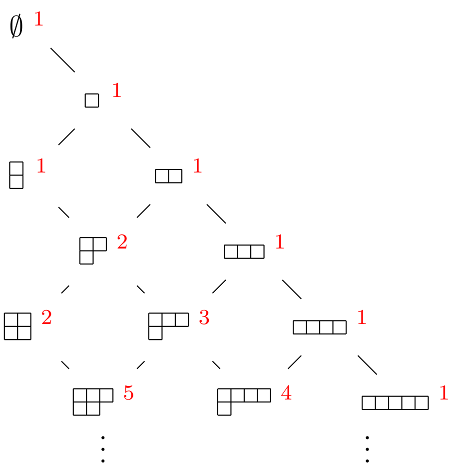
The Young lattice $\mathbb{Y}$

Brauer Bratteli diagram (add and remove)



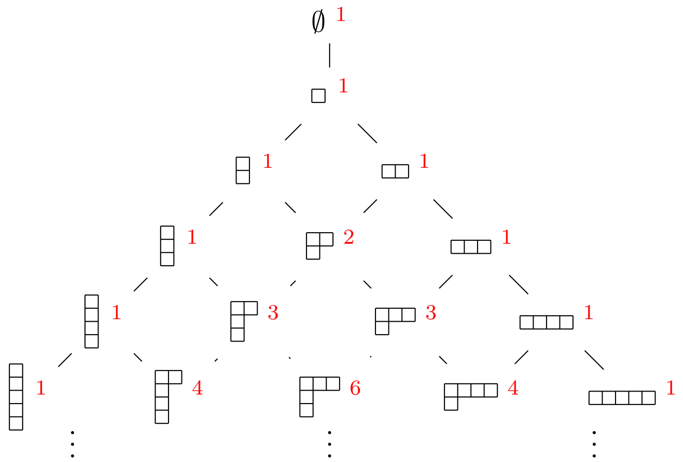
$$1^2 = 1$$
$$1^2 = 1$$
$$1^2 + 1^2 + 1^2 = 3 \cdot 1$$
$$3^2 + 1^2 + 2^2 + 1^2 = 5 \cdot 3 \cdot 1$$
$$3^2 + 6^2 + 6^2 + 1^2 + 3^2 + 2^2 + 3^2 + 1^2 = 7 \cdot 5 \cdot 3 \cdot 1$$

Temperley-Lieb Bratteli diagram (restrict to two rows)



$$1^2 = 1$$
$$1^2 = \tfrac{1}{2}\binom{2}{1}$$
$$1^2 + 1^2 = \tfrac{1}{3}\binom{4}{2}$$
$$2^2 + 1^2 = \tfrac{1}{4}\binom{6}{3}$$
$$2^2 + 3^2 + 1^2 = \tfrac{1}{5}\binom{8}{4}$$
$$5^2 + 4^2 + 1^2 = \tfrac{1}{6}\binom{10}{5}$$

Pascal's triangle (restrict to along the wall)



$$1^2 = \binom{0}{0}$$

$$1^2 + 1^2 = \binom{2}{1}$$

$$1^2 + 2^2 + 1^2 = \binom{4}{2}$$

$$1^2 + 3^2 + 3^2 + 1^2 = \binom{6}{3}$$

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = \binom{8}{4}$$

21.1.1 Partitions

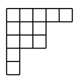
A *partition* is $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\ell \in \mathbb{Z}_{\geq 0}$, $\lambda_1, \ldots, \lambda_\ell$ and $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$.

A *box* is an element of $\mathbb{Z}^2$.

Identify a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with a set of boxes

$$\lambda = \{(r, c) \in \mathbb{Z} \times \mathbb{Z} \mid r \in \{1, \ldots, \ell\} \text{ and } c_r \in \{1, \ldots, \lambda_r\} \},$$

so that λ has λ_r boxes in row r.

$\lambda = (53311) =$  $= \left\{ \begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), \\ (2,1), & (2,2), & (2,3), & & \\ (3,1), & (3,2), & (3,3), & & \\ (4,1), & & & & \\ (5,1) & & & & \end{matrix} \right\}$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ let

$$\ell(\lambda) = \ell \qquad \text{and} \qquad |\lambda| = \lambda_1 + \cdots + \lambda_\ell.$$

Write

$$\lambda \subseteq \mu \quad \text{if } \lambda \text{ is a subset of } \mu \qquad \text{(as a collection of boxes).}$$

The *conjugate* of λ is

$$\lambda' = \{(c, r) \mid (r, c) \in \lambda\}.$$

For $n \in \mathbb{Z}_{\geq 0}$ let

$$\mathbb{Y}_n = \{\text{partitions } \lambda \text{ with } |\lambda| = n\} \qquad \text{and} \qquad \mathbb{Y} = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Y}_n.$$

21.1.2 Standard tableaux

Let $\lambda \in \mathbb{Y}_n$ and identify λ with the set of boxes of λ. A *standard tableau of shape* λ is a function $T\colon \lambda \to \{1, \ldots, n\}$ such that

(a) If $(r, c), (r, c+1) \in \lambda$ then $T(r, c) < T(r, c+1)$.

(b) If $(r, c), (r+1, c) \in \lambda$ then $T(r, c) < T(r+1, c)$.

Identify a standard tableau T with the filling of the boxes of λ which places $T(r, c)$ in the box (r, c). For example,

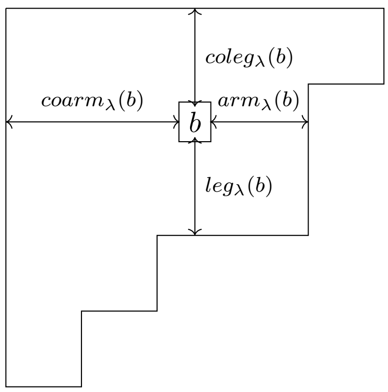
1	3	4	8	11
2	6	7	9	
5	10			

is a standard tableau of shape $\lambda = (542)$.

Let

$$\hat{S}_n^\lambda = \{\text{standard tableaux of shape } \lambda\} \qquad \text{and} \qquad f_\lambda = \mathrm{Card}(\hat{S}_n^\lambda).$$

If $(r, c) \in \lambda$ define



$$\mathrm{coleg}_\lambda(b) = r - 1,$$

$$\mathrm{coarm}_\lambda(b) = c - 1, \qquad b = (r, c), \qquad \mathrm{arm}_\lambda(b) = \lambda_r - c,$$

$$\mathrm{leg}_\lambda(b) = \lambda'_c - r.$$

Theorem 21.1. *Let $n \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{Y}_n$ and let f_λ be the number of standard tableaux of shape λ. Then*

$$f_\lambda = \frac{n!}{\prod_{b \in \lambda}(\mathrm{arm}_\lambda(b) + \mathrm{leg}_\lambda(b) + 1)} \qquad \textit{and} \qquad n! = \sum_{\mu \in \mathbb{Y}_n} f_\mu^2.$$

21.1.3 Exercises

1. Give an example of two partitions λ and μ such that $\lambda \subseteq \mu$, and two partitions γ and β such that $\gamma \not\subseteq \beta$. In each case represent the partitions both as a sequence and pictorially, as a collection of boxes.

2. For a partition λ let
$$R(\lambda) = \sum_{b \in \lambda} (\mathrm{coarm}_\lambda(b) - \mathrm{coleg}_\lambda(b)).$$
   (a) Find partitions $\lambda \neq \mu$ such that $|\lambda| = |\mu|$ and $R(\lambda) = R(\mu)$.
   (b) Do some internet searching to get a feel for how this question is related to Jucys-Murphy elements.

3. For a partition λ let
$$n(\lambda) = \sum_{b \in \lambda} (\mathrm{coleg}_\lambda(b) - 1).$$

(a) Find partitions $\lambda \neq \mu$ such that $|\lambda| = |\mu|$ and $n(\lambda) = n(\mu)$.

(b) Do some internet searching to get a feel for how this question is related to Springer fibers.

4. Fix $m, n \in \mathbb{Z}_{>0}$ with $m < n$ and let $(m^n) = \underbrace{(m, \ldots, m)}_{n \text{ parts}}$.

   (a) Determine the number of partitions μ such that $\mu \subseteq (m^n)$.

   (b) Do some internet searching to get a feel for how this question is related to the Grassmannian of m-planes in n-space.

5. For each $k \in \{1, 2, 3, 4, 5\}$ and each partition λ of k list the standard tableaux of shape λ. In each case explicitly verify the identities stated in Theorem 21.1.

6. Do some internet searching for what, and how many, proofs of each of the identities in Theorem 21.1 are in the literature, what their length is and what tools they use.

7. Let $n \in \mathbb{Z}_{>0}$. Give a careful description of a bijection between the set S_n^λ of standard tableaux of shape λ and the set of paths from $\emptyset$ to λ in $\mathbb{Y}$.

8. Let b_n^λ be the number of paths from $\emptyset$ to λ on level n of the Brauer Bratelli diagram.

   (a) For $n \in \{1, \ldots, 5\}$ and each partition λ calculate b_n^λ.

   (b) For each $n \in \{1, \ldots, 5\}$ calculate

$$\sum_\lambda (b_n^\lambda)^2, \quad \text{where the sum is over partition that appear on level } n.$$

   (c) Formulate an analogue of Theorem 21.1 for the Brauer Bratelli diagram.

9. Let c_n^λ be the number of paths from $\emptyset$ to λ on level n of the Temperley-Lieb Bratelli diagram.

   (a) For $n \in \{1, \ldots, 5\}$ and each partition λ calculate c_n^λ.

   (b) For each $n \in \{1, \ldots, 5\}$ calculate

$$\sum_\lambda (c_n^\lambda)^2, \quad \text{where the sum is over partition that appear on level } n.$$

   (c) Formulate an analogue of Theorem 21.1 for the Temperley-Lieb Bratelli diagram.

10. Let d_n^λ be the number of paths from $\emptyset$ to λ on level n of Pascal's triangle.

   (a) For $n \in \{1, \ldots, 5\}$ and each partition λ calculate d_n^λ.

   (b) For each $n \in \{1, \ldots, 5\}$ calculate

$$\sum_\lambda (d_n^\lambda)^2, \quad \text{where the sum is over partitions that appear on level } n.$$

   (c) Formulate an analogue of Theorem 21.1 for Pascal's triangle.

   (d) Write a careful proof of the analogue of Theorem 21.1 for Pascal's triangle that you have formulated.

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