20 Generating symmetric functions and q-binomial theorems

20.1 Generating function definitions

Define

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) \qquad \text{and} \qquad (a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots.$$

Define $g_r = g_r(x;q,t), q_r = q_r(x;t), h_r = h_r(x), e_r = e_r(x)$ by the generating functions

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} g_r z^r, \qquad \prod_{i=1}^n \frac{1-tx_iz}{1-x_iz} = \sum_{r\in\mathbb{Z}_{\geq 0}} q_r z^r,$$
$$\prod_{i=1}^n \frac{1}{1-x_iz} = \sum_{r\in\mathbb{Z}_{\geq 0}} h_r z^r, \qquad \prod_{i=1}^n (1+x_iz) = \sum_{r\in\mathbb{Z}_{\geq 0}} e_r z^r,$$

Remark 20.1. In later sections we will understand that the g_r are, up to a normalization factor, the Macdonald polynomials for a single row, the q_r are Hall-Littlewood polynomials for a single row, and the h_r are Schur functions for a single row. In formulas

$$\begin{aligned}
g_r &= \frac{(t;q)_r}{(q;q)_r} P_{(r)}(x;q,t), && \text{one row Macdonald polynomials,}\\
q_r &= (1-t)P_{(r)}(x;0,t), && \text{one row Hall-Littlewood polynomials,}\\
h_r &= s_{(r)}(x), && \text{one row Schur functions, and}\\
e_r &= P_{(1^r)}(x;q,t) && \text{one column Macdonald polynomials,}\\
&= P_{(1^r)}(x;0,t) && \text{one column Hall-Littlewood polynomials,}\\
&= s_{(1^r)}(x) && \text{one column Schur functions.}
\end{aligned}$$

□

Extend these definitions just slightly by defining $\tilde{g}_r = \tilde{g}_r(x;q,t,u)$ and $\tilde{q}_r = \tilde{q}_r(x;t,u)$ by

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{g}_r z^r \qquad \text{and} \qquad \prod_{i=1}^n \frac{1-tx_iz}{1-ux_iz} = \sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{q}_r z^r.$$

This is not really an extension since $g_r(x;q,t) = \tilde{g}_r(x;q,t,1)$ and

$$\tilde{g}_r(x_1,\ldots,x_n;q,t,u) = u^r\tilde{g}_r(u^{-1}x_1,\ldots,u^{-1}x_n;q,t,u) = u^r g_r(x;q,tu^{-1}),$$

so that any formula for $\tilde{g}_r$ immediately converts to a formula for g_r and vice versa. From the generating function definitions,

$$\begin{aligned}
\tilde{q}_r(x;t,u) &= \tilde{g}_r(x;0,t,u), & q_r(x;t) &= \tilde{g}_r(x;0,t,1),\\
h_r(x) &= \tilde{g}_r(x;0,0,1), & e_r(x) &= \tilde{g}_r(x;0,-1,0).
\end{aligned} \tag{20.1}$$

20.2 Formulas in terms of power sums

The *power sums* $p_r \in \mathbb{C}[x_1, \ldots, x_n]$, for $r \in \mathbb{Z}_{\ge 0}$, are defined by

$$p_0 = 1 \qquad \text{and} \qquad p_r = x_1^r + \cdots + x_n^r \quad \text{for } r \in \mathbb{Z}_{>0}.$$

For a sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}.$$

Since

$$\log(1-z) = \int \frac{-1}{1-z}\, dz = \int -(1 + z + z^2 + \cdots)\, dz = -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \cdots = -\sum_{r \in \mathbb{Z}_{>0}} \frac{1}{r} z^r$$

then

$$\begin{aligned}
\log\Big(\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty}\Big) &= \sum_{i=1}^n \sum_{\ell \in \mathbb{Z}_{\ge 0}} \big(\log(1 - tx_izq^\ell) - \log(1 - ux_izq^\ell)\big) \\
&= \sum_{i=1}^n \sum_{\ell \in \mathbb{Z}_{\ge 0}} \sum_{r \in \mathbb{Z}_{>0}} \big(-\frac{1}{r} t^r x_i^r q^{\ell r} z^r + \frac{1}{r} u^r x_i^r q^{\ell r} z^r \big) \\
&= \sum_{\ell \in \mathbb{Z}_{\ge 0}} \sum_{r \in \mathbb{Z}_{>0}} \frac{1}{r}(u^r - t^r) q^{\ell r} p_r z^r = \sum_{r \in \mathbb{Z}_{>0}} \Big(\frac{u^r - t^r}{1 - q^r}\Big) \frac{p_r}{r} z^r. \qquad (20.2)
\end{aligned}$$

Define

$$z_\lambda(q,t,u) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{u^{\lambda_i} - t^{\lambda_i}}, \qquad \text{where} \quad z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$$

for $\lambda = (\lambda_1, \ldots, \lambda_n) = (1^{m_1} 2^{m_2} \cdots)$. Taking the exponential of both sides of (3.2) gives

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty} = \sum_{r \in \mathbb{Z}_{\ge 0}} \Big(\sum_{|\lambda| = r} \frac{1}{z_\lambda(q,t,u)} p_\lambda(x) \Big) z^r$$

so that

$$\tilde{g}_r = \sum_{|\lambda| = r} \frac{1}{z_\lambda(q,t,u)} p_\lambda(x) = \sum_{|\lambda| = r} \Big(\prod_{i=1}^{\ell(\lambda)} \frac{u^{\lambda_i} - t^{\lambda_i}}{1 - q^{\lambda_i}} \Big) \frac{p_\lambda}{z_\lambda}. \qquad (20.3)$$

Applying (3.1) gives

$$\tilde{q}_r = \sum_{|\lambda| = r} \Big(\prod_{i=1}^{\ell(\lambda)} (u^{\lambda_i} - t^{\lambda_i}) \Big) \frac{p_\lambda}{z_\lambda}, \qquad q_r = \sum_{|\lambda| = r} \Big(\prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i}) \Big) \frac{p_\lambda}{z_\lambda},$$

$$h_r = \sum_{|\lambda| = r} \frac{1}{z_\lambda} p_\lambda(x), \qquad e_r = \sum_{|\lambda| = r} (-1)^{r - \ell(\lambda)} \frac{p_\lambda}{z_\lambda}.$$

20.3 Generalized Newton identities

Taking the coefficient of z^r on each side of the identity

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty} \prod_{i=1}^n \frac{(ux_iz;q)_\infty}{(sx_iz;q)_\infty} = \prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(sx_iz;q)_\infty}$$

gives

$$\tilde{g}_r(x;q,t,u) + \Big(\sum_{j=1}^{r-1} \tilde{g}_j(x;q,t,u)\tilde{g}_{r-j}(x;q,u,s)\Big) + \tilde{g}_r(x;q,u,s) = \tilde{g}_r(x;q,t,s). \tag{20.4}$$

Using the specializations in (3.1),

$$\tilde{q}_r(x;t,u) + \Big(\sum_{j=1}^{r-1} \tilde{q}_j(x;t,u)\tilde{q}_{r-j}(x;u,s)\Big) + \tilde{q}_r(x;u,s) = \tilde{q}_r(x;t,s),$$

$$\tilde{q}_r(x;t,u) + \left(\sum_{j=1}^{r-1} h_j(x)u^j\tilde{q}_{r-j}(x;t,u)\right) - h_r(x)(u^r-t^r) = 0,$$

$$\tilde{q}_r(x;t,u) + \left(\sum_{j=1}^{r-1} e_j(x)(-t)^j\tilde{q}_{r-j}(x;t,u)\right) + (-1)^r e_r(x)(u^r-t^r) = 0,$$

$$\sum_{j=0}^{r}(-t)^{r-j}(u^j-t^j)h_j(x)e_{r-j}(x) = (u-t)\tilde{q}_r(x;t,u),$$

$$r\tilde{q}_r(x;t,u) - \left(\sum_{j=1}^{r-1} p_j(x)(u^j-t^j)\tilde{q}_{r-j}(x;t,u)\right) - p_r(x)(u^r-t^r) = 0,$$

Further specializations give the *Wronski identities*

$$\sum_{i+j=k}(-1)^i e_i h_j = 0 \qquad \text{and} \qquad \sum_{i+j=k}(-1)^i(t^iq^j-1)e_ig_j = 0$$

and the *Newton identities*

$$kh_k = \sum_{i=1} p_i h_{k-i} \qquad \text{and} \qquad ke_k = \sum_{i=1}^k (-1)^{i-1}p_ie_{k-i}. \tag{20.5}$$

20.4 Formulas in terms of sequences $(i_1,\ldots,i_r)$

Using the geometric series expansions

$$\frac{1}{1-ux_iz} = 1 + ux_iz + u^2x_i^2z^2 + \cdots$$

gives

$$\frac{1-tx_iz}{1-ux_iz} = 1 + \frac{(u-t)x_iz}{1-ux_iz} = 1 + (u-t)x_iz(1+ux_iz+u^2x_i^2z^2+\cdots),$$

Apply this, factor by factor, to the product

$$\prod_{i=1}^n \frac{1-tx_iz}{1-ux_iz} = \Big(\frac{1-tx_1z}{1-ux_1z}\Big)\cdots\Big(\frac{1-tx_nz}{1-ux_nz}\Big)$$

to get

$$\tilde{q}_r = \sum_{1\le i_1\le\cdots\le i_r\le n} (u-t)^{1+\mathrm{Card}\{j\ |\ i_j<i_{j+1}\}} u^{\mathrm{Card}\{j\ |\ i_j=i_{j+1}\}} x_{i_1}x_{i_2}\cdots x_{i_r}. \tag{20.6}$$

Dividing $\tilde{q}_r$ by $(u-t)$ and specializing $t=u$ gives

$$\Big(\frac{1}{u-t}\tilde{q}_r\Big)\Big|_{t=u} = p_r = \sum_{i_1=i_2=\cdots=i_r} x_{i_1}\cdots x_{i_r}. \tag{20.7}$$

Applying $(1-ux_iz)^{-1} = 1 + ux_iz + u^2x_i^2z^2 + \cdots$ and expanding, factor by factor, the product

$$\prod_{i=1}^n \frac{1-tx_iz}{1-ux_iz} = \Big(\frac{1}{1-ux_1z}\Big)\cdots\Big(\frac{1}{1-ux_nz}\Big)(1-tx_nz)\cdots(1-tx_1z)$$

gives

$$\tilde{q}_r = \sum_{i_1\le i_2\le\cdots\le i_k>i_{k+1}>\cdots>i_r} u^{k-1}(-t)^{r-k} x_{i_1}\cdots x_{i_k}x_{i_{k+1}}\cdots x_{i_r}. \tag{20.8}$$

Applying (3.1) gives

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}\cdots x_{i_r}, \qquad \text{and} \qquad h_r = \sum_{i_1\le i_2\le\cdots\le i_r} x_{i_1}\cdots x_{i_r}. \tag{20.9}$$

20.5 Formulas in terms of monomial symmetric functions

For $\lambda = (\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n$ with $\lambda_1\ge\cdots\ge\lambda_n\ge 0$, the *monomial symmetric function* is defined by

$$m_\lambda = \sum_{\gamma\in S_n\lambda} x^\gamma, \qquad \text{where} \quad x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}.$$

Applying the expansion (from the infinite q-binomial theorem, see below)

$$\frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} = \sum_{r\in\mathbb{Z}_{\ge 0}} \frac{(t;q)_r}{(q;q)_r} x_i^r z^r,$$

and expanding the product

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} = \frac{(tx_1z;q)_\infty}{(x_1z;q)_\infty}\cdots\frac{(tx_nz;q)_\infty}{(x_nz;q)_\infty},$$

gives

$$\tilde{g}_r = \sum_{|\mu|=r} u^r \frac{(tu^{-1};q)_\mu}{(q;q)_\mu} m_\mu, \qquad \text{where} \quad \frac{(tu^{-1};q)_\mu}{(q;q)_\mu} = \frac{(tu^{-1};q)_{\mu_1}\cdots(tu^{-1};q)_{\mu_\ell}}{(q;q)_{\mu_1}\cdots(q;q)_{\mu_\ell}}.$$

if $\mu = (\mu_1,\ldots,\mu_\ell)$. Using the specializations in (3.1),

$$\tilde{q}_r = \sum_{|\mu|=r} u^{r-\ell(\mu)}(u-t)^{\ell(\mu)} m_\mu, \qquad g_r = \sum_{|\mu|=r} \frac{(t;q)_\mu}{(q;q)_\mu} m_\mu, \qquad q_r = \sum_{|\mu|=r} (1-t)^{\ell(\mu)} m_\mu.$$

$$h_r = \sum_{|\mu|=r} m_\mu, \qquad e_r = m_{(1^r)}, \qquad p_r = m_{(r)}. \tag{20.10}$$

20.6 The Cauchy-Macdonald kernel

For a sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ define

$$\tilde{g}_\lambda = \tilde{g}_{\lambda_1} \tilde{g}_{\lambda_2} \cdots \tilde{g}_{\lambda_\ell}, \qquad \tilde{q}_\lambda = \tilde{q}_{\lambda_1} \tilde{q}_{\lambda_2} \cdots \tilde{q}_{\lambda_\ell}, \qquad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}, \qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}.$$

Then

$$\prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(ux_iy_j;q)_\infty} = \sum_\lambda \tilde{g}_\lambda(x;q,t,u) m_\lambda(y) = \sum_\lambda \frac{1}{z_\lambda(q,t,u)} p_\lambda(x) p_\lambda(y).$$

20.7 Monomial expansion of $\tilde{g}_\lambda, \tilde{q}_\lambda, h_\lambda$ and e_λ

For a sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ define

$$\tilde{g}_\lambda = \tilde{g}_{\lambda_1} \tilde{g}_{\lambda_2} \cdots \tilde{g}_{\lambda_\ell}, \qquad \tilde{q}_\lambda = \tilde{q}_{\lambda_1} \tilde{q}_{\lambda_2} \cdots \tilde{q}_{\lambda_\ell}, \qquad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}, \qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}.$$

For an $n \times \ell$ matrix $a = (a_{ij})$ with entries from $\mathbb{Z}_{\geq 0}$ let

$$\begin{matrix} rs(a) = (\mu_1, \ldots, \mu_n), \\ cs(a) = (\lambda_1, \ldots, \lambda_\ell), \end{matrix} \qquad \text{where} \qquad \mu_i = \sum_{j=1}^{\ell} a_{ij} \quad \text{and} \quad \lambda_j = \sum_{i=1}^{n} a_{ij},$$

so that $rs(a)$ and $cs(a)$ are the sequences of row sums and column sums of a, respectively. Define

$$x^a = x^{rs(a)} = \prod_{i=1}^{n} \prod_{j=1}^{\ell} (x_i)^{a_{ij}}, \qquad y^a = y^{cs(a)} = \prod_{j=1}^{\ell} \prod_{i=1}^{n} (y_j)^{a_{ij}}, \qquad \mathrm{wt}_{q,u,t}(a) = \prod_{j=1}^{\ell} \prod_{i=1}^{n} u^{a_{ij}} \frac{(tu^{-1};q)_{a_{ij}}}{(q;q)_{a_{ij}}},$$

where, by definition, $(a;q)_0 = 1$. For a sequence $\mu = (\mu_1, \ldots, \mu_n)$ of nonnegative integers let

$$A_{\mu\lambda} = \{a \in M_{n\times\ell}(\mathbb{Z}_{\geq 0}) \mid cs(a) = \lambda,\ rs(a) = \mu\}.$$

Then

$$\tilde{g}_\lambda = \sum_\mu a_{\mu\lambda}(q,t) m_\mu, \qquad \text{where} \qquad a_{\mu\lambda}(q,t) = \sum_{a \in A_{\mu\lambda}} \mathrm{wt}_{q,t,u}(a), \tag{20.11}$$

and the first sum is over partitions μ such that $|\mu| = |\lambda|$.

20.8 Binomial theorems

Using

$$\prod_{i=1}^{n} (u + x_i z) = u^n \prod_{i=1}^{n} (1 + x_i \tfrac{z}{u}) = \sum_{r=0}^{n} u^{n-r} z^r e_r(x),$$

and

$$\prod_{i=1}^{n} \frac{1}{(u - x_i z)} = u^{-n} \prod_{i=1}^{n} \frac{1}{(1 - x_i \frac{z}{u})} = \sum_{r \in \mathbb{Z}_{\geq 0}} u^{-n-r} z^r h_r(x),$$

and specializing $x_1 = x_2 = \cdots = x_n = 1$ gives the *binomial theorem*,

$$(u+z)^n = \sum_{r=0}^{n} u^{n-r} z^r \binom{n}{r} \qquad \text{and} \qquad (u-z)^{-n} = \sum_{r \in \mathbb{Z}_{\geq 0}} u^{-n-r} z^r \binom{n+r-1}{r},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = e_r(1,1,\ldots,1) \qquad \text{and} \qquad \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!} = h_r(1,1,\ldots,1).$$

Letting $x_i = q^{i-1}$ gives the q-*binomial theorem*,

$$\prod_{i=1}^{n}(u+q^{i-1}z) = \sum_{r=0}^{n} q^{\frac{1}{2}r(r-1)} \begin{bmatrix} n \\ r \end{bmatrix} u^{n-r}z^r \qquad \text{and} \qquad \prod_{i=1}^{n} \frac{1}{(u-q^{i-1}z)} = \sum_{r\in\mathbb{Z}_{\geq 0}} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} u^{-n-r}z^r,$$

where

$$e_r(1,q,q^2\ldots,q^{n-1}) = \frac{(q;q)_n}{(q;q)_r(q;q)_{n-r}} = \begin{bmatrix} n \\ r \end{bmatrix} \qquad \text{and}$$

$$h_r(1,q,q^2\ldots,q^{n-1}) = \frac{(q;q)_{n+r-1}}{(q;q)_r(q;q)_{n-1}} = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}.$$

A general infinite q-binomial theorem is

$$\frac{(tz;q)_\infty}{(uz;q)_\infty} = \prod_{i=1}^{\infty}\Big(\frac{1-tq^{i-1}z}{1-uq^{i-1}z}\Big) = \sum_{r\in\mathbb{Z}_{\geq 0}} \Big(\prod_{i=1}^{r}\frac{u-tq^{i-1}}{1-q^i}\Big)z^r = \sum_{r\in\mathbb{Z}_{\geq 0}} u^r\frac{(tu^{-1};q)_r}{(q;q)_r}z^r, \tag{20.12}$$

A one sentence proof of the infinite q-binomial theorem: Recognize that

$$L(z;q,t,u) = \frac{(tz;q)_\infty}{(uz;q)_\infty} \qquad \text{satisfies the recursion} \qquad L(z;q,t,u) = \frac{(1-tz)}{(1-uz)}L(qz;q,t,u)$$

which provides a recursion on the coefficients of $L(z;q,tu) = \sum_{r\in\mathbb{Z}_{\geq 0}} c_r(q,t,u)z^r$ as

$$c_r(q,t,u)q^r - tc_{r-1}(q,t,u)q^{r-1} = c_r(q,t,u) - uc_{r-1}(q,t,u).$$

so that

$$c_r(q,t,u) = c_{r-1}(q,t,u)\frac{u-tq^{r-1}}{1-qq^{r-1}} = u^r\frac{(tu^{-1};q)_r}{(q;q)_r}.$$

Specializing $t=0$ and $u=0$ in (3.12) give

$$\prod_{i=1}^{\infty}(1+q^{i-1}z) = (-z,q)_\infty = \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{1\cdot q\cdot q^2\cdots q^{r-1}}{(q,q)_r}z^r \qquad \text{and}$$

$$\prod_{i=1}^{\infty}\frac{1}{(1-q^{i-1}z)} = \frac{1}{(z,q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{1}{(q,q)_r}z^r.$$

The finite q-binomial theorem is obtained from (3.12) by putting $t=q^n$ and $u=1$ so that the left hand side becomes

$$\frac{(q^nz;q)_\infty}{(z;q)_\infty} = \frac{1}{(z,q)_n} = \prod_{i=1}^{n}\frac{1}{1-q^{i-1}z}$$

and the right hand side is

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \frac{(q^n;q)_r}{(q;q)_r}z^r, \qquad \text{with} \quad (q^n;q)_r\frac{1}{(q;q)_r} = \frac{(q;q)_{n+r-1}}{(q;q)_{n-1}}\cdot\frac{1}{(q;q)_r} = \begin{bmatrix} n+r-1 \\ r \end{bmatrix},$$

so that

$$\prod_{i=1}^{n}\frac{1}{1-q^{i-1}z} = \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{(q;q)_{n+r-1}}{(q;q)_r(q;q)_{n-1}}z^r = \sum_{r\in\mathbb{Z}_{\geq 0}} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} z^r.$$

20.9 Exercises

1. Let $n = 3$. Use the series
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$
and the definitions to explicitly compute $g_r, \tilde{q}_r, q_r, h_r$, and e_r for $r \in \{1, 2, 3, 4\}$. For these cases, explicitly verify the relations in (3.1) and the formulas in terms of sequences given in §3.4.

2. For the case $n = 3$ write out p_0, p_1, p_2, p_3, p_4 and explicitly write out the expansions of $\tilde{g}_r$, $\tilde{q}_r$, q_r, h_r and e_r in terms of power sums.

3. Explicitly verify the identity (3.4) when $n = 3$ and $r \in \{1, 2, 3, 4\}$.

4. Describe exactly what specialization of (3.4) produces each of the Wronski identities and each of the Newton identities in (3.5).

5. Compute the matrix $(a_{\mu\lambda}(q,t))$ for partitions of size 1, 2, 3 and 4. For each partition with ≤ 4 boxes.explicitly verify the formula for $\tilde{g}_\lambda$ in (3.11).

6. For $n = 3$ and partitions with ≤ 4 boxes, explicitly compute the monomial symmetric functions m_μ. Explciitly verify the formulas in (3.5) for $r \in \{1, 2, 3, 4\|$.

7. Give a combinatorial proof that
$$e_r(1, 1, \ldots, 1) = \binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{and} \qquad h_r(1, 1, \ldots, 1) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}.$$

8. Give a combinatorial proof that
$$e_r(1, q, q^2 \ldots, q^{n-1}) = \frac{(q;q)_n}{(q;q)_r (q;q)_{n-r}} = \begin{bmatrix} n \\ r \end{bmatrix} \qquad \text{and}$$
$$h_r(1, q, q^2 \ldots, q^{n-1}) = \frac{(q;q)_{n+r-1}}{(q;q)_r (q;q)_{n-1}} = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}.$$

9. Foe $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3 \in \mathbb{Z}_{\ge 0}$ such that $\gamma_1 + \gamma_2 + \gamma_3 \le 4$ and $\delta_1 + \delta_2 + \delta_3 \le 4$, explcitly verify the equality of the coefficients of $x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}$ in each of the expressions in
$$\prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(ux_iy_j;q)_\infty} = \sum_\lambda \tilde{g}_\lambda(x;q,t,u) m_\lambda(y) = \sum_\lambda \frac{1}{z_\lambda(q,t,u)} p_\lambda(x) p_\lambda(y).$$

10. Use
$$\prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(ux_iy_j;q)_\infty} = \sum_\lambda \tilde{g}_\lambda(x;q,t,u) m_\lambda(y)$$
to give a combinatorial proof of (3.11).

11. Use the Wronski identites and the Newton identitities to show that
$$e_n = \det \begin{pmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ 1 & h_1 & h_2 & \cdots & h_{n-1} \\ 0 & 1 & h_1 & \cdots & h_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & h_1 \end{pmatrix}, \qquad h_n = \det \begin{pmatrix} e_1 & e_2 & e_3 & \cdots & e_n \\ 1 & e_1 & e_2 & \cdots & e_{n-1} \\ 0 & 1 & e_1 & \cdots & e_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & e_1 \end{pmatrix},$$

$$p_n = \det\begin{pmatrix} e_1 & 1 & 0 & \cdots & 0 \\ 2e_2 & e_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ ne_n & e_{n-1} & e_{n-2} & \cdots & e_1 \end{pmatrix}, \quad (-1)^{n-1}p_n = \det\begin{pmatrix} h_1 & 1 & 0 & \cdots & 0 \\ 2h_2 & h_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ nh_n & h_{n-1} & h_{n-2} & \cdots & h_1 \end{pmatrix},$$

$$n!e_n = \det\begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ p_2 & p_1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & \cdot & \cdots & n-1 \\ p_n & p_{n-1} & \cdot & \cdots & p_1 \end{pmatrix}, \quad n!h_n = \det\begin{pmatrix} p_1 & -1 & 0 & \cdots & 0 \\ p_2 & p_1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{n-1} & p_{n-2} & \cdot & \cdots & -(n-1) \\ p_n & p_{n-1} & \cdot & \cdots & p_1 \end{pmatrix}.$$

12. An $n \times n$ doubly stochastic matrix is a matrix $a \in M_n(\mathbb{R}_{\geq 0})$ with $rs(a) = (1, 1, \ldots, 1)$ and $cs(a) = (1, 1, \ldots, 1)$. Show that a matrix M is doubly stochastic if and only if M is a nonegative linear combination of permutation matrices (the Birkhoff-von Neumann theorem).

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