16 The symmetric group

16.1 Permutations and the symmetric group

Let $n \in \mathbb{Z}_{>0}$. The vector space of $n \times n$ matrices

$$M_n(\mathbb{C}) \quad \text{has } \mathbb{C}\text{-basis} \quad \{E_{ij} \mid i,j \in 1,\ldots,n\},$$

whee E_{ij} is the matrix with 1 in the (i,j) entry and 0 elsewhere.
A *permutation of* n is $w \in M_{n\times n}(\mathbb{C})$ such that

(a) There is exactly one nonzero entry in each row and each column.
(b) The nonzero entries are 1.

The *symmetric group* is the set

$$S_n = \{w \in M_{n\times n}(\mathbb{C}) \mid w \text{ is a permutation of } \{1,\ldots,n\}\}$$

with matrix multiplication. Identify a permutation $w \in M_{n\times n}(\mathbb{C})$ with a bijection $w\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ by

$$w(i) = j \qquad \text{if } w_{ji} = 1,$$

where w_{ij} is the (i,j)-entry of the matrix w.

16.2 Transpositions and simple reflections

The *transpositions*, or *reflections*, in S_n are

$$s_{ij} = 1 + E_{ij} + E_{ji} - E_{ii} - E_{jj}, \qquad \text{for } i,j \in \{1,\ldots,n\} \text{ with } i \neq j.$$

The *simple transpositions* are

$$s_1 = s_{12}, \qquad s_2 = s_{23}, \quad \ldots, \quad s_{n-1} = s_{n-1,n}.$$

16.3 Inductive structure

The *general linear group* is the set

$$GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \text{there exists } A^{-1} \in M_n(\mathbb{C}) \text{ with } AA^{-1} = 1 \text{ and } A^{-1}A = 1\}$$

with matrix multiplication.

Proposition 16.1. *The maps*

$$\begin{array}{ccc} GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) & \longrightarrow & GL_{n+m}(\mathbb{C}) \\ (A,B) & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} S_n \times S_m & \longrightarrow & S_{n+m} \\ (v,w) & \longmapsto & \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \end{array}$$

are injective group homomorphisms.

16.4 Coxeter elements

Let $\gamma_1 = E_{11}$ in S_1 and

$$\gamma_k = E_{12} + E_{23} + \cdots + E_{k-1,k} + E_{k1} \quad \text{in } S_k,$$

for $k \in \mathbb{Z}_{>1}$. A *Coxeter element* of S_n is an element of the conjugacy class of γ_n in S_n.

16.5 Young subgroups

For $\mu_1, \ldots, \mu_\ell \in \mathbb{Z}_{>0}$ let

$$\gamma_\mu = \gamma_{\mu_1} \times \cdots \times \gamma_{\mu_\ell} \quad \text{in} \quad S_{\mu_1} \times \cdots \times S_{\mu_\ell} \subseteq S_{\mu_1+\cdots+\mu_\ell}.$$

The group $S_{\mu_1} \times \cdots \times S_{\mu_\ell}$ is a *Young subgroup*, or *parabolic subgroup*, of $S_{\mu_1+\cdots+\mu_\ell}$.

16.6 Conjugacy classes

For $\mu_1, \ldots, \mu_\ell \in \mathbb{Z}_{>0}$ let $n = \mu_1 + \ldots + \mu_\ell$ and let

$$[\gamma_\mu] \quad \text{denote the conjugacy class of } \gamma_\mu \text{ in } S_n.$$

A *partition of n* is $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that $\lambda_1, \ldots, \lambda_\ell \in \mathbb{Z}_{>0}$ and $\lambda_1 \geq \cdots \geq \lambda_\ell$ and $\lambda_1 + \cdots + \lambda_\ell = n$.

Theorem 16.2.

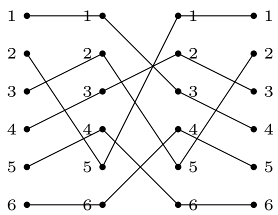
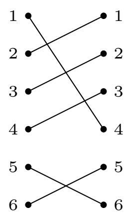
(a) The map

$$\begin{array}{ccc} \{\textit{partitions of } n\} & \longrightarrow & \{\textit{conjugacy classes of } S_n\} \\ \lambda & \longmapsto & [\gamma_\lambda] \end{array} \quad \textit{is a bijection.}$$

(b) If λ is a partition of n and m_i is the number of parts of size i (write $\lambda = (1^{m_1} 2^{m_2} \cdots)$) then

$$\mathrm{Card}([\gamma_\lambda]) = \frac{n!}{z_\lambda}, \qquad \textit{where} \quad z_\lambda = (1^{m_1} 2^{m_2} \cdots)(m_1! m_2! \cdots).$$

Proof idea. For example, if $w = (531624)$ then

$x^{-1}wx =$  $=$  $= \gamma_{42}, \qquad$ where $x = (152346)$.

If $\lambda = (4,4,3,2,2,2,1,1,1,1) = (4^2 3^1 2^3 1^4)$ then

$$\gamma_\lambda = \gamma_4 \times \gamma_4 \times \gamma_3 \times \gamma_2 \times \gamma_2 \times \gamma_2 \times \gamma_1 \times \gamma_1 \times \gamma_1 \times \gamma_1$$

and

$$\begin{aligned} z_\lambda = \mathrm{Card}(\mathrm{Stab}(\gamma_\lambda)) &= (2! \cdot 4 \cdot 4) \cdot 3 \cdot (3! \cdot 2 \cdot 2 \cdot 2) \cdot (4! \cdot 1 \cdot 1 \cdot 1 \cdot 1) \\ &= 4^2 \cdot 3^1 \cdot 2^3 \cdot 1^4 \cdot 2! \cdot 1! \cdot 3! \cdot 4! \end{aligned}$$

so that

$$\mathrm{Card}([\gamma_\lambda]) = \frac{\mathrm{Card}(S_n)}{\mathrm{Card}(\mathrm{Stab}(\gamma_\lambda))} = \frac{n!}{z_\lambda}.$$

□

16.6.1 Exercises

1. Let $n \in \mathbb{Z}_{>0}$ and let $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{Z}^r_{>0}$ such that $\mu_1 + \cdots + \mu_r = n$.
   (a) Prove carefully that $|S_n| = n!$.
   (b) Prove carefully that $S_{\mu_1} \times \cdots \times S_{\mu_r}$ is a subgroup of S_n.
   (b) Prove carefully that $GL_{\mu_1}(\mathbb{C}) \times \cdots \times GL_{\mu_r}(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$.
   (c) Determine $|S_{\mu_1} \times \cdots \times S_{\mu_r}|$.
2. Let $n \in \mathbb{Z}_{>0}$.
   (a) Prove that if $w \in S_n$ then w can be written as a product of simple transpositions.
   (b) Explain the relation between this question and row reduction for matrices.
3. Explicitly list the Coxeter elements for the symmetric groups S_5, S_4, S_3, S_2 and S_1. Display these as permutation matrices, as bijections, and as products of simple transpositions.
4. Carefully define group, group action, stabilizer, orbit and 'set of orbit representatives'. Use the action of S_5 on itself by conjugation to give illustrative examples of each of these terms.
5. Explicitly display the results of Theorem 16.2 for the symmetric groups S_5, S_4, S_3, S_2 and S_1.
6. Let $n \in \mathbb{Z}_{>0}$ and let $s_1, \ldots, s_{n-1}$ be the simple transpositions in S_n.
   (a) Prove that if $i \in \{1, \ldots, n-1\}$ then $s_i^2 = 1$.
   (b) Prove that if $i, j \in \{1, \ldots, n-1\}$ and $j \notin \{i-1, i+1\}$ then $s_i s_j = s_j s_i$.
   (c) Prove that if $i \in \{1, \ldots, n-2\}$ then $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.
   (d) Do some internet searching to determine the relation between this question and Coxeter groups.
7. Let $n \in \mathbb{Z}_{>0}$.
   (a) Prove carefully that S_n is a subgroup of $GL_n(\mathbb{C})$.
   (b) Prove that there are exactly two group homomorphisms $f\colon S_n \to GL_1(\mathbb{C})$. Determine these homomorphisms explicitly. Determine their images and their kernels.
8. Give a careful proof of Proposition 16.1.
9. Let G be a group acting on a set X. For $x \in X$ let Gx be the orbit of x and let G_x be the stabilizer of x.
   (a) Prove carefully that $X = \bigcup_{x \in X} Gx$.
   (b) Prove that if $x, y \in X$ then $Gx = Gy$ or $Gx \cap Gy = \emptyset$.
   (c) Show that G_x is a subgroup of G, carefully define the G-set G/G_x and prove that
   $$Gx \cong G/G_x, \qquad \text{as } G\text{-sets}.$$
   (d) Assume that X and G are finite. Prove carefully that $|G_x| \cdot |Gx| = |G|$.
   (e) For part (c), it is not actually necessary to assume that X and G are finite. State carefully what $|G_x| \cdot |Gx| = |G|$ means when X and G are not necessarily finite and prove it carefully.

10. Give a careful proof of Theorem 16.2.

11. Let $n \in \mathbb{Z}_{>0}$.

    (a) Carefully define 'signed permutation' and prove carefully that

    $$O_n(\mathbb{Z}) = \{A \in M_n(\mathbb{C}) \mid AA^t = 1\}$$

    is the group of signed permutations.

    (b) Generalize all the definitions and results of this section to the group of signed permutations.

    (c) Do some internet searching to get a feel for how this question is related to the Weyl groups of type B and of type C.

12. Let $n \in \mathbb{Z}_{>0}$.

    (a) Prove carefully that

    $$SO_n(\mathbb{Z}) = \{A \in M_n(\mathbb{C}) \mid AA^t = 1 \text{ and } \det(A) = 1\}$$

    is the group of signed permutations with an even number of signs.

    (b) Generalize all the definitions and results of this section to the group of signed permutations with an even number of signs.

    (c) Do some internet searching to get a feel for how this question is related to the Weyl group of type D.

13. Let $n \in \mathbb{Z}_{>0}$.

    (a) Prove carefully that S_n is a subgroup of $GL_n(\mathbb{C})$.

    (b) Prove that there are exactly two group homomorphisms $f\colon S_n \to GL_1(\mathbb{C})$. Determine these homomorphisms explicitly. Determine their images and their kernels.

    (c) Explicitly determine all group homomorphism $f\colon GL_n(\mathbb{C}) \to GL_1(\mathbb{C})$. Determine their images and their kernels.

    (d) Explain why it would be more useful/comfortable if question (c) was stated as: Explicitly determine all rational group homomorphism $f\colon GL_n(\mathbb{C}) \to GL_1(\mathbb{C})$. (In other words, carefully state the difference between the answers to (c) and (d) for the case $n = 1$.)

14. Let $n \in \mathbb{Z}_{>0}$ and let $\chi\colon S_n \to \mathbb{C}$ be a function that satisfies

    $$\text{if } u, v \in S_n \text{ then } \quad \chi(uv) = \chi(vu). \qquad (*)$$

    (a) Show that χ is completely determined by the values $\{\chi(\gamma_\mu) \mid \mu \text{ is a partition of } n\}$.

    (b) Let $f\colon S_n \to GL_{732}(\mathbb{C})$ be a group homomorphism and define

    $$\chi\colon S_n \to \mathbb{C} \qquad \text{by} \quad \chi(u) = \mathrm{tr}(f(u).$$

    Show that χ satsifies the condition in (*).

    (c) Do some internet searching to get a feel for how this question is related to *characters of representations of the symmetric group.*

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