18 The subspace lattice $\mathbb{G}(\mathbb{F}^n)$

18.1 The rank generating function of $\mathbb{G}(\mathbb{F}^n)$

Let $\mathbb{F}$ be a field. The *subspace lattice* of $\mathbb{F}^n$ is

$$\mathbb{G}(\mathbb{F}^n) = \{\mathbb{F}\text{-subspaces of } \mathbb{F}^n\} \qquad \text{partially ordered by inclusion.}$$

Then $\mathbb{G}(\mathbb{F}^n)$ is a ranked modular lattice

$$\mathbb{G}(\mathbb{F}^n) = \bigsqcup_{k=0}^{n} \mathbb{G}(\mathbb{F}^n)_k, \qquad \text{where} \quad \mathbb{G}(\mathbb{F}^n)_k = \{\mathbb{F}\text{-subspaces } V \subseteq \mathbb{F}^n \text{ with } \dim(V) = k\}.$$

Theorem 18.1. *Let $\mathbb{F}_q$ be a finite field with q elements. For $r \in \mathbb{Z}_{>0}$ let*

$$[r] = \frac{q^r - 1}{q-1}, \qquad [r]! = [r][r-1]\cdot[2][1], \qquad \text{and let} \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!},$$

for $k \in \{0, 1, \ldots, n\}$. Then

$$\mathrm{Card}(\mathbb{G}(\mathbb{F}_q^n)_k) = \begin{bmatrix} n \\ k \end{bmatrix}$$

and

$$\sum_{k=0}^{n} x^k q^{\frac{1}{2}k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix} = (1+x)(1+xq)\cdots(1+xq^{n-1}) = (-x;q)_n.$$

18.2 Automorphisms of $\mathbb{G}(\mathbb{F}^n)$

For a poset P, let $\mathrm{Aut}(P)$ denote the group of automorphism of P.

Proposition 18.2. *Let $\mathbb{F}$ be a field. As groups,*

$$Aut(\mathbb{G}(\mathbb{F}^n)) \cong GL_n(\mathbb{F}), \qquad \text{where } GL_n(\mathbb{F}) = \{g \in M_n(\mathbb{F}) \mid g^{-1} \text{ exists in } M_n(\mathbb{F})\}.$$

18.3 Projective space and cosets

Let $\mathbb{F}$ be a field and define an equivalence relation on $\mathbb{F}^n - \{(0,\ldots,0)\}$ by

$$[a_1,\ldots,a_n] = [\lambda a_1,\ldots,\lambda a_n], \qquad \text{if } a_1,\ldots,a_n \in \mathbb{F} \text{ and } \lambda \in \mathbb{F}^\times.$$

The *projective space* $\mathbb{P}^{n-1}$ is

$$\mathbb{P}^{n-1} = \{\text{equivalence classes}\}.$$

Let $\{e_1,\ldots,e_n\}$ be an $\mathbb{F}$-basis of $\mathbb{F}^n$ and let

$$E = (0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = \mathbb{F}^n), \qquad \text{where} \quad E_k = \mathbb{F}\text{-span}\{e_1,\ldots,e_k\},$$

for $k \in \{0,\ldots,n\}$. Let $B = \{b \in GL_n(\mathbb{F}) \mid b \text{ is upper triangular}\}$ and

$$P_k = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathrm{GL}_n(\mathbb{F}) \mid A \in M_{k\times k}(\mathbb{F}),\ B \in M_{k\times(n-k)}(\mathbb{F}),\ C \in M_{(n-k)\times n-k)}(\mathbb{F}) \right\}$$

for $k \in \{1,\ldots,n\}$.

Proposition 18.3. *Let*

$$G = GL_n(\mathbb{F}^n) = \mathrm{Aut}(\mathbb{G}(\mathbb{F}^n)) \ \textit{acting on}\ \mathbb{G}(\mathbb{F}^n)$$

and let

$$\mathcal{F}(\mathbb{G}(\mathbb{F}^n)) = \{\textit{maximal chains in}\ \mathbb{G}(\mathbb{F}^n)\}.$$

(a)

$$\mathrm{Stab}_G(E_k) = P_k \qquad \textit{and} \qquad \mathrm{Stab}_G(E) = B.$$

(b)

$$\mathbb{G}(\mathbb{F}^n)_k \cong G/P_k \qquad \textit{and} \qquad \mathcal{F}(\mathbb{G}(\mathbb{F}^n)) \cong G/B,$$

(c)

$$\mathbb{G}(\mathbb{F}^n)_1 \cong \mathbb{P}^{n-1} \quad \textit{and} \quad \mathbb{G}(\mathbb{F}^n)_{n-1} \cong \mathbb{P}^{n-1} \quad \textit{and} \quad \mathbb{G}(\mathbb{F}^n)_k \cong \mathbb{G}(\mathbb{F}^n)_{n-k}.$$

18.4 Counting

Let $\mathbb{F}_q$ be a finite field with q elements.

Proposition 18.4. *Let*

$$G = GL_n(\mathbb{F}^n) = \mathrm{Aut}(\mathbb{G}(\mathbb{F}^n))$$

and let B be the subgroup of upper triangular matrices in G.

(a)

$$\mathrm{Card}(GL_n(\mathbb{F}_q^n)) = [n]!q^{\frac{1}{2}n(n-1)}(q-1)^n, \qquad \mathrm{Card}(B) = q^{\frac{1}{2}n(n-1)}(q-1)^n,$$

and

$$\mathrm{Card}(\mathcal{F}(\mathbb{G}(\mathbb{F}_q^n))) = [n]!.$$

(b)

$$\mathrm{Card}(\mathbb{P}^1) = 1+q, \qquad \textit{and} \qquad \mathrm{Card}(\mathbb{P}^{n-1}) = 1+q+\cdots+q^{n-1}.$$

18.5 The Hecke algebra

Let $\mathbb{C}[G/B]$ be the $\mathbb{C}$-vector space with basis indexed by the elements of $\mathcal{F}(\mathbb{G}(\mathbb{F}_q^n))$. The group $G = GL_n(\mathbb{F}_q)$ acts on $\mathbb{C}[G/B]$ by the $\mathbb{C}$-linear maps given by

$$g(0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = (0 \subsetneq gV_1 \subsetneq \cdots \subsetneq gV_n), \qquad \text{for } g \in GL_n(\mathbb{F}_q).$$

The following result produces linear operators $T_1, \ldots, T_{n-1}$ that all commute with the acion of $GL_n(\mathbb{F}_q)$ on $\mathbb{C}[G/B]$.

Theorem 18.5. *For $i \in \{1, \ldots, n-1\}$ define a $\mathbb{C}$-linear map $T_i \colon \mathbb{C}[G/B] \to \mathbb{C}[G/B]$ by*

$$T_i(0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{V_{i-1} \subsetneq W \subsetneq V_{i+1}} (0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_{i+1} \subsetneq \cdots \subsetneq V_n).$$

Then

$$T_j^2 = (q-1)T_j + q, \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad T_jT_k = T_kT_j, \qquad gT_j = T_jg$$

for $i \in \{1, \ldots, n-2\}$ and $j, k \in \{1, \ldots, n-1\}$ with $k \notin \{j-1, j+1\}$ and $g \in GL_n(\mathbb{F}_q)$.

18.6 Exercises

1. (a) Carefully draw the Hasse diagram of $\mathbb{G}(\mathbb{F}_2^3)$.
   (b) Explicitly verify the results of Theorem 18.1 in the case of $\mathbb{G}(\mathbb{F}_2^3)$.
   (b) Carefully draw the picture of the Fano plane and explain how the Fano plane encodes the lattice $\mathbb{G}(\mathbb{F}_2^3)$.
   (c) Write down all the elements of $GL_3(\mathbb{F}_2^3)$ and of its subgroup B of upper triangular elements. Explicitly verify the results of Proposition 18.4 for this case.
   (d) Explicitly display action of each element of $GL_3(\mathbb{F}_2)$ on the maximal chains in $\mathbb{G}(\mathbb{F}_2^3)$.
   (e) Explicitly compute the action of T_1 and T_2 from Theorem 18.5 on $\mathbb{C}[G/B]$ for the case $\mathbb{G}(\mathbb{F}_2^3)$).
2. Prove that $\mathbb{G}(\mathbb{F}^n)$ is a ranked modular lattice.
3. Provide a careful proof of Theorem 18.1.
4. Provide a careful proof of Proposition 18.2.
5. Provide a careful proof of Proposition 18.3.
6. Provide a careful proof of Proposition 18.4.
7. Provide a careful proof of Theorem 18.5.

References

[AAR] G. Andrews, R. Askey, R. Roy, *Special functions*, Encyclopedia Math. Appl. **71** Cambridge University Press 1999. xvi+664 pp. ISBN:0-521-62321-9 ISBN:0-521-78988-5 MR1688958

[BS17] Bump, Daniel and Schilling, Anne, *Crystal bases. Representations and combinatorics*, World Scientific 2017. xii+279 pp. ISBN:978-981-4733-44-1 MR3642318.

[HH08] D. Flath, T. Halverson, K. Herbig, *The planar rook algebra and Pascal's triangle*, Enseign. Math. (2) **55** (2009) 77 - 92, MR2541502, arXiv:0806.3960.

[HR95] T. Halverson and A. Ram, Characters of algebras containing a Jones basic construction: the Temperley-Lieb, Okada, Brauer, and Birman-Wenzl algebras Adv. Math. **116** (1995) 263-321, MR1363766s

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford Mathematical Monographs, Oxford University Press, New York, 1995. ISBN: 0-19-853489-2, MR1354144. 11.4, 11.8, 11.10

[Mac03] I.G. Macdonald, *Affine Hecke Algebras and Orthogonal Polynomials*, Cambridge Tracts in Mathematics **157** Cambridge University Press, Cambridge, 2003. MR1976581.

[Nou23] M. Noumi, *Macdonald polynomials – Commuting family of q-difference operators and their joint eigenfunctions*, Macdonald polynomials—commuting family of q-difference operators and their joint eigenfunctions. Springer Briefs in Mathematical Physics **50** Springer 2023 viii+132 pp. ISBN:978-981-99-4586-3 ISBN:978-981-99-4587-0 MR4647625

[St86] R.P. Stanley, *Enumerative combinatorics,*
Volume 1, Second edition Cambridge Stud. Adv. Math. **49** Cambridge University Press, Cambridge 2012 xiv+626 pp. ISBN:978-1-107-60262-5 MR2868112
Enumerative combinatorics. Vol. 2, Second edition Cambridge Stud. Adv. Math. **208** Cambridge University Press 2024, xvi+783 pp. ISBN:978-1-009-26249-1, ISBN:978-1-009-26248-4, MR4621625

[Weh90] K.H. Wehrhahn, *Combinatorics. An Introduction*, Undergraduate Lecture Notes in Mathematics **1** Carslaw publications 1990, 162 pp. ISBN 18753990038