19 The subset lattice $\mathbb{S}(n)$

The *subset lattice* $\mathbb{S}(n)$ is

$$\mathbb{S}(n) = \{\text{subsets of } \{1,\ldots,n\}\}, \qquad \text{partially ordered by inclusion.}$$

The subset lattice $\mathbb{S}(n)$ is a ranked modular lattice with

$$\mathbb{S}(n) = \bigsqcup_{k=0}^{n} \mathbb{S}(n)_k, \qquad \text{wtih} \quad \mathbb{S}(n)_k = \{\text{subsets } V \subseteq \{1,\ldots,n\} \text{ with } \mathrm{Card}(V) = k\}.$$

Then

$$\mathrm{Card}(\mathbb{S}(n)_k) = \binom{n}{k} \qquad \text{and} \qquad \sum_{k=0}^{n} x^k \binom{n}{k} = (1+x)^n \qquad \text{(Sncounts)}$$

is the rank generating function for $\mathbb{S}(n)$.

Proposition 19.1. *The automorphism group of* $\mathbb{S}(n)$ *is the symmetric group*

$$\mathrm{Aut}(\mathbb{S}(n)) = S_n.$$

Proposition 19.2. *For* $k \in \{1,\ldots,n\}$ *let* $E_k = \{1,\ldots,k\}$. *Then* $E_k \in \mathbb{S}(n)$ *and* $(\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_n)$ *is a maximal chain in* $\mathbb{S}(n)$.

$$\mathrm{Stab}_{S_n}(E_k) = S_k \times S_{n-k} \qquad \textit{and} \qquad \mathrm{Stab}_{S_n} = \{1\} = S_1 \times \cdots \times S_1.$$

$$\mathbb{S}(n)_k \cong \frac{S_n}{S_k \times S_{n-k}} \qquad \textit{and} \qquad \mathcal{F}(\mathbb{S}(n)) \cong S_n/\{1\} = S_n.$$

$$\mathbb{S}(n)_k \cong \mathbb{S}(n)_{n-k}.$$

19.1 Maximal chains in $\mathbb{S}(n)$

Proposition 19.3. *The map*

$$\begin{array}{ccc} \mathcal{F}(\mathbb{S}(n)) & \longrightarrow & S_n \\ (\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) & \longmapsto & (V_1, V_2 - V_1, \ldots, V_n - V_{n-1}) \end{array} \qquad \textit{is a bijection.}$$

Let $\mathbb{C}S_n$ be the vector space with basis indexed by the elements of $\mathcal{F}(\mathbb{S}(n))$. For $i \in \{1,\ldots,n-1\}$ define a $\mathbb{C}$-linear transformation $s_i \colon \mathbb{C}S_n \to \mathbb{C}S_n$ by

$$s_i(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{V_{i-1} \subsetneq W \subsetneq V_{i+1}} (\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_{i+1} \subsetneq \cdots \subsetneq V_n).$$

Then

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_i s_j = s_j s_i \ \text{ if } j \notin \{i-1, i+1\}, \qquad \text{(sirels)}$$

and

$$s_i g = g s_i, \qquad \text{for } g \in S_n, \qquad \text{(sipastg)}$$

where

$$g(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = (\emptyset \subsetneq gV_1 \subsetneq \cdots \subsetneq gV_n) \qquad \text{for } g \in S_n \text{ and } (\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) \in \mathcal{F}(\mathbb{S}(n)).$$

19.2 Simple reflections

Let S_n be the symmetric group of permutation matrices and let

$$s_i = 1 + E_{i,i+1} + E_{i+1,i} - E_{ii} - E_{i+1,i+1}, \qquad \text{for } i \in \{1, \ldots, n-1\}.$$

The following theorem shows that the symmetric group S_n is a Coxeter group.

Theorem 19.4. *The symmetric group S_n is presented by generators $s_1, \ldots, s_{n-1}$ and relations*

$$s_j^2 = 1, \qquad s_i s_{i s+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_j s_k = s_k s_j,$$

for $j, k \in \{1, \ldots, n-1\}$ with $k \notin \{j-1, j+1\}$ and $i \in \{1, \ldots, n-2\}$.

Proof sketch. The proof requires four steps:

(1) Generators A in terms of generators B.
(2) Generators B in terms of generators A.
(3) Relations A from relations B.
(4) Relations B from relations A.

Here

Generators A: { permutation matrices}

Relations A: { matrix multiplication of permutation matrices}

Generators B: { simple transpositions}

Relations B: { the braid relations in the statement }

□

19.3 Length and reduced words

Let $w \in S_n$. A reduced word for w is an expression $w = s_{i_1} \cdots s_{i_\ell}$ with $i_1, \ldots, i_\ell \in \{1, \ldots, n-1\}$ and ℓ minimal.

$$\text{The } \textit{length} \text{ of } w \text{ is } \ell(w), \text{ the length of a reduced word for } w.$$

Proposition 19.5. *Let*

$$\mathrm{Inv}(w) = \{(i,j) \mid i, j \in \{1, \ldots, n\} \text{ with } i < j \text{ and } w(i) > w(j)\}.$$

Then

$$\ell(w) = \mathrm{Card}(\mathrm{Inv}(w)).$$

19.4 A reduced word algorithm

Let $w \in S_n$. The following is an explicit algorithm for producing a reduced word for w.
Let $j_1 > 1$ be minimal such that $w_{j,1} \neq 0$. If j_1 does not exist set $w^{(1)} = w$ and if j_1 does exist set

$$w^{(1)} = s_1 \cdots s_{j_1 - 1} w.$$

Let $j_2 > 2$ be minimal such that $w^{(1)}_{j,2} \neq 0$. If j_2 does not exist set $w^{(2)} = w^{(1)}$ and if j_2 does exist set

$$w^{(2)} = s_2 \cdots s_{j_2 - 1} w^{(1)}.$$

Continue this process to produce $w^{(1)}, \ldots, w^{(n)}$. Then $w^{(n)} = 1$ and

$$w = \cdots (s_{j_2-1} \cdots s_2)(s_{j_1-1} \cdots s_1) \quad \text{is a reduced word for } w. \qquad \text{(favredwd)}$$

19.5 The graph of reduced words

Define a graph $\Gamma(w)$ with

Vertices:	$\{\text{reduced words of } w\}$
Edges:	$u \to u'$ if $u' = s_{i_1} \cdots s_{i_\ell}$ is obtained from $u = s_{j_1} \cdots s_{j_\ell}$ by applying a relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ or a relation $s_i s_j = s_j s_i$ with $j \notin \{i-1, i+1\}$.

Theorem 19.6. *Let $w \in S_n$. The graph $\Gamma(w)$ of reduced words of w is connected.*

Proof. Let

$$w = s_{i_1} \cdots s_{i_\ell} \qquad \text{and} \qquad w = s_{j_1} \cdots s_{j_\ell}$$

be reduced words.
Case 1: $i_1 = j_1$. The two reduced words for w have the same first letter. By induction, the reduced words $v = s_{i_2} \cdots s_{i_\ell}$ and $v = s_{j_2} \cdots s_{j_\ell}$ are connected.
Case 2: $i_1 \neq j_1$. Since $\ell(s_{j_1 w}) < \ell(w)$ then there exists k such that $s_{j_1} w = s_{i_1} \cdots s_{i_{k-1}} \cancel{s_{i_k}} s_{i_{k+1}} \cdots s_{i_\ell}$.
Case 2a: $k \neq \ell$. Then

$$\begin{aligned} w &= s_{j_1} \cdots s_{j_\ell} \\ w &= s_{j_1} s_{i_1} \cdots s_{i_{k-1}} \cancel{s_{i_k}} s_{i_{k+1}} \cdots s_{i_\ell} \quad \text{and} \\ w &= s_{i_1} \cdots s_{i_\ell} \end{aligned}$$

are all reduced words for w. Since the first factor is the same in the first two of these they are connected. Since the last factor is the same in the last two of these they are connected. So, by transitivity, the first is connected to the last.
Case 2b: $k = \ell$ and $j_1 \notin \{i_1 - 1, i_1 + 1\}$. Then

$$\begin{aligned} w &= s_{j_1} \cdots s_{j_\ell}, \\ w &= s_{j_1} s_{i_1} \cdots s_{i_{\ell-1}}, \\ w &= s_{i_1} s_{j_1} \cdots s_{i_{\ell-1}} \quad \text{and} \\ w &= s_{i_1} s_{i_2} \cdots s_{i_\ell} \end{aligned}$$

and the first two are connected since they have the same first letter, the middle two are connected by the move $s_{j_1} s_{i_1} = s_{j_1} s_{i_1}$ and the last two are connected since they have the same first letter.

Case 2c: $k = \ell$ and $j_1 \in \{i_1 - 1, i_1+\}$. Then

$$\begin{aligned}
w &= s_{i_1} s_{i_2} \cdots s_{i_\ell}, \\
w &= s_{i_1} s_{j_1} s_{i_1} \cdots s_{i_{r-1}} \cancel{s_{i_r}} s_{i_{r+1}} \cdots s_{i_{\ell-1}}, \\
w &= s_{j_1} s_{i_1} s_{j_1} \cdots s_{i_{r-1}} \cancel{s_{i_r}} s_{i_{r+1}} \cdots s_{i_{\ell-1}}, \quad \text{and} \\
w &= s_{j_1} s_{j_2} \cdots s_{j_\ell},
\end{aligned}$$

and the first two are connected since they have the same first letter, the middle two are connected by the move $s_{i_1} s_{j_1} s_{i_1} = s_{j_1} s_{i_1} s_{j_1}$ and the last two are connected since they have the same first letter. □

19.6 Exercises

1. Carefully prove that $\mathbb{S}(n)$ is a ranked modular lattice.
2. Prove the identities in (Sncounts).
3. Give a careful proof of Proposition 19.1.
4. Give a careful proof of Proposition 19.3.
5. Prove the identities in (sirels) and (sipastg).
6. Give a complete, thorough, beautifully exposited proof of Theorem 19.4
7. Give a careful proof of Proposition 19.5.
8. Pick 5 random elements of S_5 (use random.org to do this) and use the algorithm from Section 19.4 to explcitly compute a reduced word for each of them. For each of these elements also compute the inversion set $\mathrm{Inv}(w)$.
9. Let $w = (4321) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ in S_4. Carefully draw the graph of reduced words $\Gamma(w)$.
10. Let
$$a = s_1 s_2 s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1 \qquad \text{and} \qquad b = s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_1 s_2 s_1.$$
    (a) Show that these are both reduced words for the same permutation w_0 in S_5 and determine w explicitly.
    (b) Use the steps of the proof of Theorem 19.6 to construct an explicit path from a to b in $\Gamma(w)$.

References

[AAR] G. Andrews, R. Askey, R. Roy, *Special functions*, Encyclopedia Math. Appl. **71** Cambridge University Press 1999. xvi+664 pp. ISBN:0-521-62321-9 ISBN:0-521-78988-5 MR1688958

[BS17] Bump, Daniel and Schilling, Anne, *Crystal bases. Representations and combinatorics*, World Scientific 2017. xii+279 pp. ISBN:978-981-4733-44-1 MR3642318.

[HH08] D. Flath, T. Halverson, K. Herbig, *The planar rook algebra and Pascal's triangle*, Enseign. Math. (2) **55** (2009) 77 - 92, MR2541502, arXiv:0806.3960.

[HR95] T. Halverson and A. Ram, Characters of algebras containing a Jones basic construction: the Temperley-Lieb, Okada, Brauer, and Birman-Wenzl algebras Adv. Math. **116** (1995) 263-321, MR1363766s

[Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second edition, Oxford Mathematical Monographs, Oxford University Press, New York, 1995. ISBN: 0-19-853489-2, MR1354144. 11.4, 11.8, 11.10

[Mac03] I.G. Macdonald, *Affine Hecke Algebras and Orthogonal Polynomials*, Cambridge Tracts in Mathematics **157** Cambridge University Press, Cambridge, 2003. MR1976581.

[Nou23] M. Noumi, *Macdonald polynomials – Commuting family of q-difference operators and their joint eigenfunctions*, Macdonald polynomials—commuting family of q-difference operators and their joint eigenfunctions. Springer Briefs in Mathematical Physics **50** Springer 2023 viii+132 pp. ISBN:978-981-99-4586-3 ISBN:978-981-99-4587-0 MR4647625

[St86] R.P. Stanley, *Enumerative combinatorics,*
Volume 1, Second edition Cambridge Stud. Adv. Math. **49** Cambridge University Press, Cambridge 2012 xiv+626 pp. ISBN:978-1-107-60262-5 MR2868112
Enumerative combinatorics. Vol. 2, Second edition Cambridge Stud. Adv. Math. **208** Cambridge University Press 2024, xvi+783 pp. ISBN:978-1-009-26249-1, ISBN:978-1-009-26248-4, MR4621625

[Weh90] K.H. Wehrhahn, *Combinatorics. An Introduction*, Undergraduate Lecture Notes in Mathematics **1** Carslaw publications 1990, 162 pp. ISBN 18753990038