17 Posets and lattices

17.1 Two examples

(1) Let $n \in \mathbb{Z}_{>0}$. The *subset lattice of* $\{1, \ldots, n\}$ is

$$\mathbb{S}(n) = \{\text{subsets of } \{1, \ldots, n\}\} \qquad \text{partially ordered by inclusion.}$$

(2) The *Young lattice* is

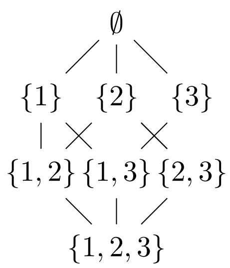
$$\mathbb{Y} = \{\text{partitions}\} \qquad \text{partially ordered by inclusion.}$$

Then

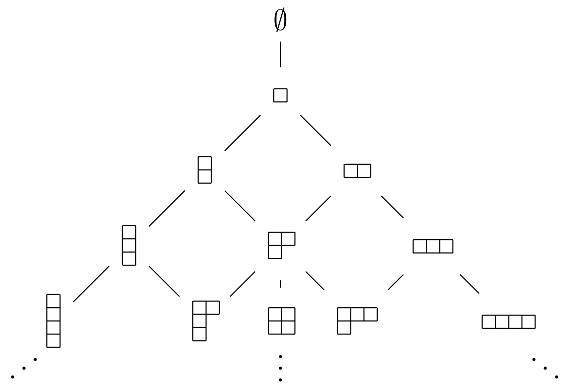
$$\mathbb{S}(n) = \bigsqcup_{k=0}^{n} \mathbb{S}(n)_k, \qquad \text{where} \quad \mathbb{S}(n) = \{\text{subsets of } \{1, \ldots, n\} \text{ with cardinality } k\},$$

and

$$\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n, \qquad \text{where} \quad \mathbb{Y}_n = \{\text{partitions with } n \text{ boxes}\}.$$



The subset lattice $\mathbb{S}(3)$



The Young lattice $\mathbb{Y}$

17.2 Posets

Let S be a set. A *relation on* S is a subset of $S \times S$.

$$\text{Write} \qquad x \le y \qquad \text{if } (x,y) \text{ is in the relation } \le.$$

A *partially ordered set*, or *poset*, is a set P with a relation $\le$ on P such that

(a) If $x \in P$ then $x \le x$,

(b) If $x, y, z \in P$ and $x \le y$ and $y \le z$ then $x \le z$, and

(c) If $x, y \in P$ and $x \le y$ and $y \le x$ then $x = y$.

The *Hasse diagram* of P is the graph with

$$\text{Vertices: } P \qquad \text{and} \qquad \text{Directed edges: } x \to y \text{ if } x \le y.$$

A *maximal chain in* P is a sequence $x_1, \ldots, x_n \in P$ with $n \in \mathbb{Z}_{>0}$ such that

(a) $x_1 < x_2 < \cdots < x_n$,

(b) If $i \in \{1, \ldots, n-1\}$ then there does not exist $y \in P$ such that $x_i < y < x_{i+1}$,

(c) There does not exist $y \in P$ such that $y < x_1$,

(d) There does not exist $y \in P$ such that $x_n < y$.

17.3 Morphisms and automorphisms

A *morphism of posets* is a function $f\colon P \to Q$ such that P and Q are posets and

$$\text{if } x, y \in P \text{ and } x \le y \quad \text{then} \quad f(x) \le f(y).$$

An *isomorphism* of posets is a morphism $f\colon P \to Q$ such that

$$\text{the inverse function } f^{-1}\colon Q \to P \text{ exists and } f^{-1} \text{ is a morphism of posets.}$$

A *automorphism* of P is an isomorphism $f\colon P \to P$ of posets.

17.4 Lattices

Let P be a poset and $E \subseteq P$.
An *upper bound of* E *in* S is an element $b \in S$ such that if $y \in E$ then $y \le b$.
A *lower bound of* E *in* S is an element $l \in S$ such that if $y \in E$ then $y \ge b$.
The *infimum*, or *greatest lower bound of* E *in* P is an element $\ell \in P$ such that

(a) If $p \in E$ then $\ell \le p$,

(b) If $m \in P$ and m is a lower bound of E in S then $m \le \ell$.

The *supremum*, or *least upper bound of* E *in* P is an element $\gamma \in P$ such that

(a) If $p \in E$ then $p \ge \gamma$,

(b) If $\tau \in P$ and τ is an upper bound of E in S then $\gamma \le \tau$.

A *lattice* is a poset P such that

$$\text{if } x, y \in P \text{ then } \inf(x,y) \text{ and } \sup(x,y) \text{ exist in } P.$$

17.5 Modular lattices

Let P be a lattice. Use the notation

$$x \wedge y = \inf(x,y) \qquad \text{and} \qquad x \vee y = \sup(x,y),$$

and the language is $x \wedge y$ is "x meet y" and $x \vee y$ is "x join y".

A *modular lattice* is a lattice P such that

$$\text{if } m,n,p \in P \text{ and } p \le m \quad \text{then} \quad m \vee (n \wedge p) = (m \vee p) \wedge n.$$

For the following results, if the term $\mathbb{Z}$-algebra and A-module are unfamiliar, there is nothing lost in replacing A by a field $\mathbb{F}$, V by an $\mathbb{F}$-vector space and A-submodules by $\mathbb{F}$-subspaces.

Theorem 17.1. *Let A be a $\mathbb{Z}$-algebra and let V be an A-module. Let*

$$\mathbb{G}(V) = \{A\text{-submodules of } V\} \qquad \text{partially ordered by inclusion.}$$

Then $\mathbb{G}(V)$ is a modular lattice.

Proposition 17.2. *Let A be a $\mathbb{Z}$-algebra and let V be an A-module. Let $M,N,P \in \mathbb{G}(V)$.*

(a) (infimums exist)

$$\inf(M,N) = M \cap N = \{v \in V \mid v \in M \text{ and } v \in N\}.$$

(b) (supremums exist)

$$\sup(M,N) = M + N = \{m+n \mid m \in M \text{ and } n \in N\}.$$

(c) (modular law)

$$\text{If } P \subseteq M \quad \text{then} \quad M + (N \cap P) = (M+) \cap P.$$

(d) (modular property)

$$\frac{M+N}{M} \cong \frac{N}{M \cap N}.$$

17.6 Exercises

1. (a) Give an example of a morphism $f\colon P \to Q$ of finite posets that is bijective and is not an isomorphism of posets. Draw a picture of the Hasse diagrams of P and Q to illustrate.
   (b) Let P be a poset and let $\mathrm{Aut}(P)$ be the set of automorphims of P. Show that $\mathrm{Aut}(P)$ with product given by composition of functions is a group.
   (c) Explicitly determine $\mathrm{Aut}(\mathbb{S}(3))$ and $\mathrm{Aut}(\mathbb{Y})$.

2. Carefully define the terms 'relation', 'equivalence relation' and 'partition of a set', and prove that the data of an equivalence relation is the same as the data of a partition of a set. In other words, define a map from equivalence relations on S to partitions of S and prove carefully that your definition is actually a function (i.e. is 'well defined') and that the function is a bijection.

3. The definition of maximal chains as given requires every maximal chain to be finite. Make an alternate/better careful defnition that does not require a maximal chain to be finite.

4. (a) Give an example of a poset P and a subset E of P such that $\sup(E)$ does not exist.
   (b) Prove that if P is a poset and E is a subset of P and $\sup(E)$ exists then $\sup(E)$ is unique.
   (c) Explain the connection between this question and the 'universal property' of a 'universal object'.

5. Let S be a poset and let E be a subset of S.
   (a) Formulate careful definitions of minimal element of E, maximal element of E, smallest element of E and largest element of E.
   (b) Give an example where E has two maximal elements.
   (c) Prove that if E has a largest element then it is unique.
   (d) How do the maximal element and the largest element of E relate to $\sup(E)$?
   (e) Explain the relation between the terms maximal element and largest element and the terms maximum, local maximum and global maximum and explain how they apply to the setting of maxima and minima that one meets in a first year calculus class.

6. (Ranked posets) Let $n \in \mathbb{Z}_{\geq 0}$. A *ranked poset of rank* n is a poset P such that there exists a unique function $\mathrm{rk}\colon P \to \{0, 1, \ldots, n\}$ such that

   (RPa) If x is a minimal element of P then $\mathrm{rk}(x) = 0$,

   (RPb) If $x, y \in P$ and $x < y$ and there does not exist $z \in P$ such that $x < z < y$ then $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$.

   Prove that a poset P is a ranked poset of rank n if and only if every maximal chain in P has length n.

7. (Totally ordered sets) Let S be a set. A *total order* on S is a partial order $\leq$ such that

   $$\text{If } x, y \in S \text{ then } x \leq y \text{ or } y \leq x.$$

   A *totally ordered set* is a set S with a total order $\leq$ on S.

   A poset S is *well ordered* if S satisfies: every nonempty subset E of S has a smallest element.

   Prove that if S is well ordered then S is totally ordered.

8. (Totally ordered fields) An *ordered field* is a field $\mathbb{F}$ with a total order such that

(OFa) If $a, b, c \in \mathbb{F}$ and $a \le b$ then $a + c \le b + c$.

(OFb) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $ab \ge 0$.

(a) Carefully define the standard partial order on $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\ge 0}$, $\mathbb{Q}$ and $\mathbb{R}$.

(b) Prove that $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\ge 0}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are totally ordered sets which satisfy the conditions (OFa) and (OFb).

(c) Prove that if $\mathbb{F}$ is an ordered field then the following hold:

(1) If $a \in \mathbb{F}$ and $a > 0$ then $-a < 0$.
(2) If $a \in \mathbb{F}$ and $a \ne 0$ then $a^2 = 0$.
(3) $1 \ge 0$.
(4) If $a \in \mathbb{F}$ and $a > 0$ then $a^{-1} > 0$.
(5) If $a, b \in \mathbb{F}$ and $a \ge 0$ and $b \ge 0$ then $a + b \ge 0$.
(6) If $a, b \in \mathbb{F}$ and $0 < a < b$ then $b^{-1} < a^{-1}$.
(7) If $x, y \in \mathbb{F}$ with $x \ge 0$ and $y \ge 0$ then $x \le y$ if and only if $x^2 < y^2$.

(d) Prove that there does not exist a total order on $\mathbb{C}$ such that (OFa) and (OFb) hold.

(e) Let $p \in \mathbb{Z}_{>0}$ be prime and let $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$. Prove that there does not exist a total order on $\mathbb{F}$ such that (OFa) and (OFb) hold.

9. Let S be a poset and let E be a subset of S. The set E is *bounded in S* if E has both an upper bound and a lower bound in S. Carefully define $\mathbb{R}$ and the standard partial order on $\mathbb{R}$ and prove carefully that if $E \subseteq \mathbb{R}$ and E is bounded then $\inf(E)$ and $\sup(E)$ exist in $\mathbb{R}$.

10. Let S be a poset. The *intervals in S* are the sets

$$\begin{array}{ll} S_{[a,b]} = \{x \in S \mid a \le x \le b\}, & S_{(a,b)} = \{x \in S \mid a < x < b\}, \\ S_{[a,b)} = \{x \in S \mid a \le x < b\}, & S_{(a,b]} = \{x \in S \mid a < x \le b\}, \\ S_{(-\infty,b]} = \{x \in S \mid x \le b\}, & S_{[a,\infty)} = \{x \in S \mid a \le x\}, \\ S_{(-\infty,b)} = \{x \in S \mid x < b\}, & S_{(a,\infty)} = \{x \in S \mid a < x\}, \end{array}$$

for $a, b \in S$.

Let E be a subset of $\mathbb{R}$. Carefully define what it means for E to be a connected subset of $\mathbb{R}$ and then prove carefully that E is connected if and only if E is an interval in $\mathbb{R}$.

11. (Dedekind cuts) Let S be a poset. A *lower order ideal of S* is a subset E of S such that

$$\text{if } y \in E \text{ and } x \in S \text{ and } x \le y \quad \text{then} \quad x \in E.$$

Let R be the set of lower order ideals in $\mathbb{Q}$. Show that $R \cong \mathbb{R}$.

12. (ideals in $\mathbb{Z}$) Let

$$\mathbb{G}(\mathbb{Z}) = \{n\mathbb{Z} \mid n \in \mathbb{Z}_{\ge 0}\}, \qquad \text{partially ordered by inclusion.}$$

(a) Let $P = \mathbb{G}(\mathbb{Z}) - \{\mathbb{Z}\}$. Determine the maximal elements of P.

(b) Let $k \in \mathbb{Z}_{>0}$. Determine $\mathbb{G}(\mathbb{Z})_{\ge k}$. Draw pictures of the Hasse diagram of $\mathbb{G}(\mathbb{Z})_{\ge k}$ for $k \in \{5, 12, 48, 60, 100\}$.

(c) Let $a, b \in \mathbb{Z}_{\ge 0}$. Carefully define $\gcd(a,b)$ and $\mathrm{lcm}(a,b)$ and prove carefully that

$$\sup(a\mathbb{Z}, b\mathbb{Z}) = g\mathbb{Z} \text{ and } \inf(a\mathbb{Z}, b\mathbb{Z}) = m\mathbb{Z}, \quad \text{where } g = \gcd(a,b) \text{ and } m = \mathrm{lcm}(a,b).$$

13. Give an example of a lattice that is not modular.

14. Give a careful proof of Theorem 18.2.

15. Give a careful proof of Proposition 18.1.

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