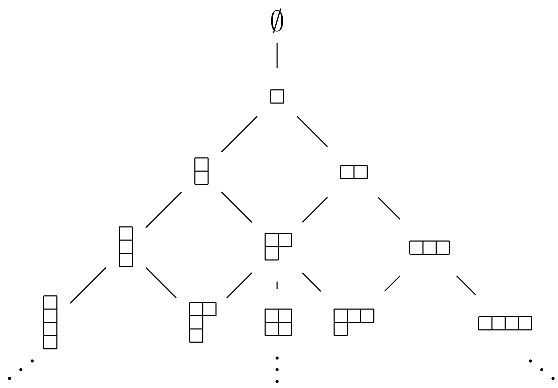
# 14 Bratelli diagrams, Partitions and the Young lattice

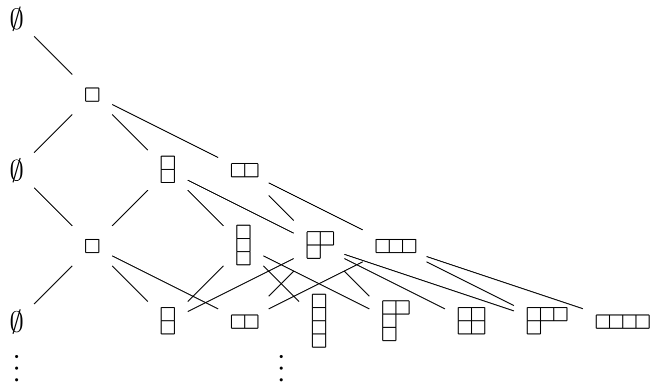
## 14.1 Bratelli diagrams

**Young's lattice** (boxes in a corner)

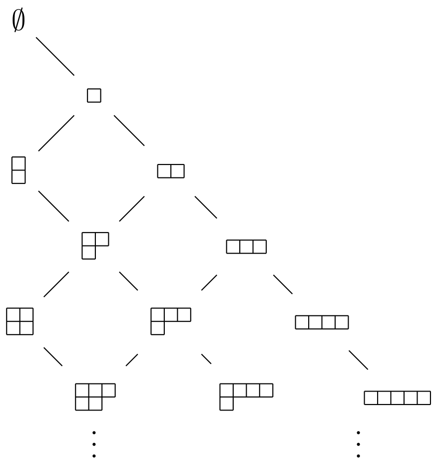


The Young lattice $\mathbb{Y}$

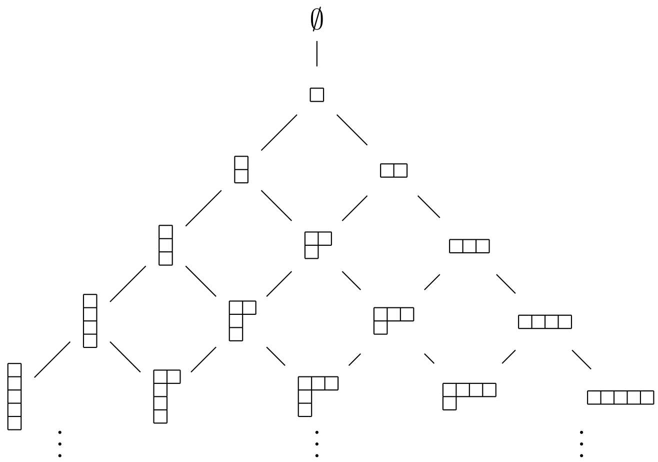
**Brauer Bratteli diagram** (add and remove)



**Temperley-Lieb Bratteli diagram** (restrict to two rows)



**Pascal's triangle** (restrict to along the wall)



## 14.2 Partitions

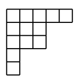
A *partition* is $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\ell \in \mathbb{Z}_{\geq 0}$, $\lambda_1, \ldots, \lambda_\ell$ and $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$.

A *box* is an element of $\mathbb{Z}^2$.

Identify a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with a set of boxes

$$\lambda = \{(r, c) \in \mathbb{Z} \times \mathbb{Z} \mid r \in \{1, \ldots, \ell\} \text{ and } c_r \in \{1, \ldots, \lambda_r\}\ \},$$

so that $\lambda$ has $\lambda_r$ boxes in row $r$.

$\lambda = (53311) =$  $= \left\{ \begin{array}{lllll} (1,1), & (1,2), & (1,3), & (1,4), & (1,5) \\ (2,1), & (2,2), & (2,3), & & \\ (3,1), & (3,2), & (3,3), & & \\ (4,1), & & & & \\ (5,1), & & & & \end{array} \right\}$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ let

$$\ell(\lambda) = \ell \qquad \text{and} \qquad |\lambda| = \lambda_1 + \cdots + \lambda_\ell.$$

Write

$$\lambda \subseteq \mu \quad \text{if } \lambda \text{ is a subset of } \mu \qquad \text{(as a collection of boxes).}$$

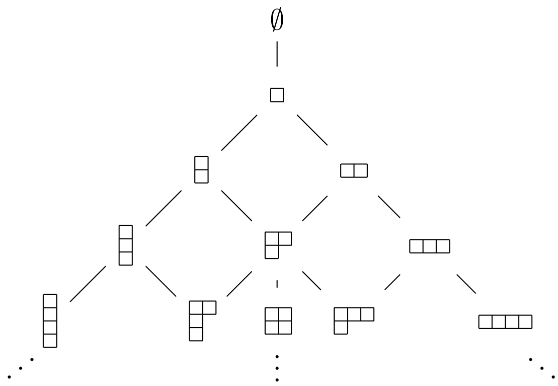
The *conjugate* of $\lambda$ is

$$\lambda' = \{(c, r) \mid (r, c) \in \lambda\}.$$

### 14.3 The Young lattice

For $n \in \mathbb{Z}_{\geq 0}$ let

$$\mathbb{Y}_n = \{\text{partitions } \lambda \text{ with } |\lambda| = n\} \qquad \text{and} \qquad \mathbb{Y} = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Y}_n.$$




The Young lattice $\mathbb{Y}$

Let $\lambda \in \mathbb{Y}_n$ and identify $\lambda$ with the set of boxes of $\lambda$.
A *standard tableau of shape* $\lambda$ is a function $T: \lambda \to \{1, \ldots, n\}$ such that

(a) If $(r, c), (r, c+1) \in \lambda$ then $T(r, c) < T(r, c+1)$.

(b) If $(r, c), (r+1, c) \in \lambda$ then $T(r, c) < T(r+1, c)$.

Identify at standard tableau of shape $\lambda$ with a path from $\emptyset$ to $\lambda$ in $\mathbb{Y}$.

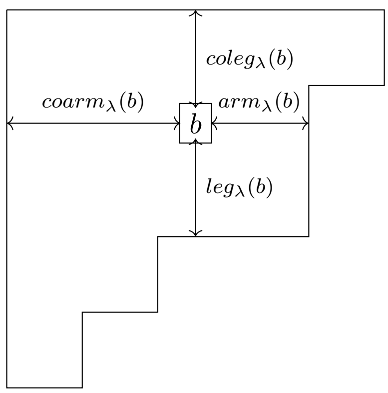
Let

$$\hat{S}_n^\lambda = \{\text{standard tableaux of shape } \lambda\} \qquad \text{and} \qquad f_\lambda = \text{Card}(\hat{S}_n^\lambda).$$

If $(r, c) \in \lambda$ define

$$\text{arm}(r, c) = \lambda_r - r,$$

$$\text{leg}(r, c) = \lambda'_c - c.$$




**Theorem 14.1.** *Let $n \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{Y}_n$. Then*

$$f_\lambda = \frac{n!}{\prod_{b \in \lambda}(\text{arm}(b) + \text{leg}(b) + 1)} \qquad \textit{and} \qquad n! = \sum_{\lambda \in \mathbb{Y}_n} f_\lambda^2.$$

## 14.4 Exercises

1. Give an example of two partitions $\lambda$ and $\mu$ such that $\lambda \subseteq \mu$, and two partitions $\gamma$ and $\beta$ such that $\gamma \not\subseteq \beta$. In each case represent the partitions both as a sequence and pictorially, as a collection of boxes.

2. For a partition $\lambda$ let
$$R(\lambda) = \sum_{b\in\lambda} (\mathrm{coarm}_\lambda(b) - \mathrm{coleg}_\lambda(b)).$$
   (a) Find partitions $\lambda \neq \mu$ such that $|\lambda| = |\mu|$ and $R(\lambda) = R(\mu)$.
   (b) Do some internet searching to get a feel for how this question is related to Jucys-Murphy elements.

3. For a partition $\lambda$ let
$$n(\lambda) = \sum_{b\in\lambda} (\mathrm{coleg}_\lambda(b) - 1).$$
   (a) Find partitions $\lambda \neq \mu$ such that $|\lambda| = |\mu|$ and $n(\lambda) = n(\mu)$.
   (b) Do some internet searching to get a feel for how this question is related to Springer fibers.

4. Fix $m, n \in \mathbb{Z}_{>0}$ with $m < n$ and let $(m^n) = \underbrace{(m, \ldots, m)}_{n \text{ parts}}$.
   (a) Determine the number of partitions $\mu$ such that $\mu \subseteq (m^n)$.
   (b) Do some internet searching to get a feel for how this question is related to the Grassmannian of $m$-planes in $n$-space.

5. For each $k \in \{1, 2, 3, 4, 5\}$ and each partition $\lambda$ of $k$ list the standard tableaux of shape $\lambda$. In each case explicitly verify the identities stated in Theorem 14.1.

6. Do some internet searching for what, and how many, proofs of each of the identities in Theorem 14.1 are in the literature, what their length is and what tools they use.

7. Let $n \in \mathbb{Z}_{>0}$. Give a careful description of a bijection between the set $S_n^\lambda$ of standard tableaux of shape $\lambda$ and the set of paths from $\emptyset$ to $\lambda$ in $\mathbb{Y}$.

8. Let $b_n^\lambda$ be the number of paths from $\emptyset$ to $\lambda$ on level $n$ of the Brauer Bratelli diagram.
   (a) For $n \in \{1, \ldots, 5\}$ and each partition $\lambda$ calculate $b_n^\lambda$.
   (b) For each $n \in \{1, \ldots, 5\}$ calculate
$$\sum_\lambda (b_n^\lambda)^2, \quad \text{where the sum is over partition that appear on level } n.$$
   (c) Formulate an analogue of Theorem 14.1 for the Brauer Bratelli diagram.

9. Let $c_n^\lambda$ be the number of paths from $\emptyset$ to $\lambda$ on level $n$ of the Temperley-Lieb Bratelli diagram.
   (a) For $n \in \{1, \ldots, 5\}$ and each partition $\lambda$ calculate $c_n^\lambda$.
   (b) For each $n \in \{1, \ldots, 5\}$ calculate
$$\sum_\lambda (c_n^\lambda)^2, \quad \text{where the sum is over partition that appear on level } n.$$

(c) Formulate an analogue of Theorem 14.1 for the Temperley-Lieb Bratelli diagram.

10. Let $d_n^\lambda$ be the number of paths from $\emptyset$ to $\lambda$ on level $n$ of Pascal's triangle.

(a) For $n \in \{1, \ldots, 5\}$ and each partition $\lambda$ calculate $d_n^\lambda$.

(b) For each $n \in \{1, \ldots, 5\}$ calculate

$$\sum_\lambda (d_n^\lambda)^2, \quad \text{where the sum is over partitions that appear on level } n.$$

(c) Formulate an analogue of Theorem 14.1 for Pascal's triangle.

(d) Write a careful proof of the analogue of Theorem 14.1 for Pascal's triangle that you have formulated.

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