15 The binomial theorem and the exponential

15.1 Binomial coefficients

For $k \in \mathbb{Z}_{\geq 0}$ define

$$0! = 1 \quad \text{and} \quad k! = k(k-1)\cdots 3\cdot 2\cdot 1, \quad \text{for } k \in \mathbb{Z}_{>0}.$$

For $k \in \{0, 1, \ldots, n\}$ define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem 15.1. *Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$.*

(a) Let S be a set with cardinality n. Then
$\binom{n}{k}$ is the number of subsets of S with cardinality k.

(b) $\binom{n}{k}$ is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$.

(c) If $k \in \{1, \ldots, n-1\}$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad \text{and} \qquad \binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{n} = 1.$$

(d) In $\mathbb{C}[x, y]$,

$$(x+y)^n = \sum_{k=0} \binom{n}{k} x^k y^{n-k}.$$

$$\begin{array}{ccccccccc} &&&& \binom{0}{0} &&&& \\ &&& \binom{1}{0} && \binom{1}{1} &&& \\ && \binom{2}{0} && \binom{2}{1} && \binom{2}{2} && \\ & \binom{3}{0} && \binom{3}{1} && \binom{3}{2} && \binom{3}{3} & \\ \binom{4}{0} && \binom{4}{1} && \binom{4}{2} && \binom{4}{3} && \binom{4}{4} \end{array} \quad = \quad \begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 1 && 3 && 3 && 1 & \\ 1 && 4 && 6 && 4 && 1 \end{array}$$

Proposition 15.2. *For a partition λ, let f_λ denote the number of standard tableaux of shape λ. Then*

$$f_{(k,1^{n-k})} = \binom{n}{k}, \qquad \sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

15.2 Formal power series

The ring of *formal power series* is

$$\mathbb{C}[[x]] = \{a_0 + a_1x + a_2x^2 + \cdots \mid a_i \in \mathbb{C}\}$$

and its field of fractions is the *ring of expressions*,

$$\mathbb{C}((x)) = \{a_{-\ell}x^{-\ell} + a_{-\ell+1}x^{-\ell+1} + a_{-\ell+2}x^{-\ell+2} + \cdots \mid \ell \in \mathbb{Z}, a_i \in \mathbb{C}\},$$

and the ring of polynomials is

$$\mathbb{C}[x] = \{a_0 + a_1x + a_2x^2 + \cdots \mid a_i \in \mathbb{C} \text{ and all but a finite number of the } a_i \text{ are } 0\}.$$

15.3 The exponential

The *exponential* is

$$\exp(x) = e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots .$$

This is the most important expression in mathematics.

The following theorem establishes the most important properites of $\exp(x)$.

Theorem 15.3.

(a) If $xy = yx$ then $\exp(x+y) = \exp(x)\exp(y)$.

(b) $\dfrac{d}{dx}\exp(x) = \exp(x)$.

The following theorem characterizes $\exp(x)$ in *two different ways.*

Theorem 15.4.

(a) If $p \in \mathbb{C}[[[x]]$ and $p(x+y) = p(x)p(y)$ then

$$\textit{there exists} \quad a \in \mathbb{C} \qquad \textit{such that} \qquad p(x) = \exp(ax).$$

(b) If $p \in \mathbb{C}[[x]]$ and $\frac{dp}{dx} = p$ then

$$\textit{there exists} \quad c_0 \in \mathbb{C} \qquad \textit{such that} \qquad p(x) = c_0\exp(x).$$

15.4 The binomial theorem

Let

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) \qquad \text{and} \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1).$$

The *q-hypergeometric series* ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}\; ; q, z\right] = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(a_0;q)_k(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k} z^k.$$

and is a q-analogue of *the generalized hypergeometric series*

$${}_{r+1}F_r\left[\begin{matrix} \alpha_0, \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_r \end{matrix}\; ; z\right] = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(\alpha_0)_k(\alpha_1)_k\cdots(\alpha_r)_k}{(1)_k(\beta_1)_k\cdots(\beta_r)_k} z^k,$$

If $\alpha \in \mathbb{Z}_{>0}$ then

$$(\alpha)_k = \frac{(\alpha+k-1)!}{(\alpha-1)!} \qquad \text{so that} \quad n! = (1)_n \quad \text{and} \quad \binom{n}{k} = \frac{(k)_{n-k}}{(1)_k}$$

when $n, k \in \mathbb{Z}_{>0}$ with $k \leq n$.

Theorem 15.5. *Let $\alpha \in \mathbb{C}$. Then*

$$(1-z)^{-\alpha} = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(\alpha)_k}{k!} z^k = \sum_{k\in\mathbb{Z}_{\geq 0}} \binom{-\alpha}{k}(-z)^k = {}_1F_0[\alpha; z]$$

Proof. (One option) Taylor series:

$$\frac{1}{k!}\frac{d^k}{dx^k}(1+x)^\alpha\Big]_{x=0} = \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k!}.$$

□

15.5 Exercises

1. Give a careful proof of Theorem 15.1.
2. Give a careful proof of Proposition 15.2. Look up Halverson-Herbig arxiv:0806.3960 to get a feel for how this question is related to the planar rook monoid algebra.
3. Provide careful proofs of Theorems 15.3 and 15.4.
4. (a) Prove that $\mathbb{C}[[x]]$ is an integral domain and that $\mathbb{C}((x))$ is the field of fractions of $\mathbb{C}[[x]]$.
   (b) Show that $\mathbb{C}[x]$ and $\mathbb{C}[[x]]$ and $\mathbb{C}((x))$ are all $\mathbb{C}$-vector spaces and describe a basis of each.
5. (a) Give a careful description of the addition and multiplication in
   $$\mathbb{C}[[x]] = \{a_0 + a_1 x + a_2 x^2 + \cdots \mid a_i \in \mathbb{C}\}$$
   (b) Give a careful description of the addition and multiplication in
   $$\mathbb{R}_{[0,10]} = \{a_0 + a_1\big(\tfrac{1}{10}\big) + a_2\big(\tfrac{1}{10}\big)^2 + \cdots \mid a_i \in \{0,1,\ldots,9\}\}.$$
   (c) Technically, there is one additional condition needed in the definition of $\mathbb{R}_{[0,10]}$. What is it?
6. Show that
   $${}_{r+1}F_r\left[\begin{matrix}\alpha_0,\alpha_1,\ldots,\alpha_r\\ \beta_1,\ldots,\beta_r\end{matrix};z\right] = \lim_{q\to 1}\left({}_{r+1}\phi_r\left[\begin{matrix}q^{\alpha_0},q^{\alpha_1},\ldots,q^{\alpha_r}\\ q^{\beta_1},\ldots,q^{\beta_r}\end{matrix};q,z\right]\right).$$
7. Prove a q-analogue of Theorem 15.5: namely,
   $${}_1\phi_0\left[\begin{matrix}a\\ \cdot\end{matrix};q,z\right] = \sum_{i=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty}$$
8. Prove that ${}_1F_0\left[\begin{matrix}\alpha\\ \cdot\end{matrix};z\right]$ satisfies the differential equation
   $$\frac{dF}{dz} = -\frac{a}{z}F$$
   and ${}_1\phi_0\left[\begin{matrix}a\\ \cdot\end{matrix};q,z\right]$ satisfies the difference equation
   $$f(qz) = \frac{1-z}{1-az}f(z).$$
9. (Gauss hypergeometric series)
   Prove that ${}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};z\right]$ satisfies the differential equation
   $$z(z-1)\frac{d^2F}{dz^2} + (c-(a+b-1)z)\frac{dF}{dz} - abF = 0.$$
   and that ${}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,z\right]$ satisfies the difference equation
   $$(q^{a+b}z - q^{c-1})\varphi(q^2z) = (-(q^a+q^b)z + q^{c-1}+1)\varphi(qz) + (z-1)\varphi(z) = 0.$$
10. Prove the identities
    $$\frac{1}{(z;q)_\infty} = \sum_{k=0}^{\infty}\frac{1}{(q;q)_k)}z^k \qquad \text{and} \qquad (z;q)_\infty = \sum_{k=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}z^k.$$

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