Draft materials for Advanced Discrete Math under construction

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Abstract

Thiese are noes in preparation for teaching Advanced Discrete Mathematics MAST90030 at University of Melbourne in 2025.

Key words— symmetric functions, crystals, binomial theorems, and hypergeometric functions

Contents

AMS Subject Classifications: Primary 05E05; Secondary 20G99.

1 Introduction

1.1 The background

I've been working as a researcher in advanced discrete mathematics for about 35 years (since 1988). A few years ago I was shocked to discover that we have a course with the title Advanced Discrete Mathematics here in the mathematics department of University of Melbourne. I had never mentally registered it because I am officially in the Pure Mathematics section of the department and this course is part of the Mathematical Physics section of the department.

The discovery of this course got me thinkng. Even though I've worked in this field for a long time and know a few things about it, I have never taught such a course, ever, in my whole career. I thought to myself: perhaps this is something that I would enjoy teaching and perhaps I could make a positive contribution for our students – I think, actually, maybe, I would like to teach course this once before I retire. So, I put my hand up, and listed it as a teaching preference for the following year's teaching allocations.

It didn't happen right away, but in the natural progression of administrative cycles, the emperors stirred up and reshuffled the ministry a bit, as they do, and I found myself assigned to teach this course in 2025. So I thought I'd think about it more realistically. What, exactly, would I like to show the students, in my one chance to teach this course? I've had a very stimulating research career in this field, with constant amazement and awe while tending the beautiful structures that we study, and I have one chance to show a few students our wondrous garden. How should I design the garden tour?

1.2 The plan

Not suprisingly, once I actually started thinking about it, I realised that I'd have to give assignments and a final exam. For this I'd need to cook up some problems, for which I'd would enjoy "covering the content" behind those problems in my in class lectures.

So, the reality is that I need to decide what problems I will solve for them in class. So I started making some lists of problams, to get a feel, for myself, of what is "out there".

1.3 Possible assignments

Assignment 1: Write an introduction to Catalan numbers, including their definition, a closed, formula (with proof), recursion relations (with proof), and a formula for their generating function (with proof), and the fact that they count noncrossing matchings on an even number of vertices (with proof). You will be marked 40% on the quality and readability of your mathematical writing, 30% on your presentation, delivery and formatting, 20% on thoroughness and 5% on whether the answers are correct.

Assignment 2: Write a careful exposition of the proof of the Coxeter presentation of the symmetric group. Be sure to include definitions (of the symmetric group and the simple transpositions), state the theorem carefully, and write a clear, complete, careful proof of the theorem. You will be marked 40% on the quality and readability of your mathematical writing, 30% on your presentation, delivery and formatting, 20% on thoroughness and 5% on whether the answers are correct.

Assignment 3: Write up a careful check of the tables on pages 111 and 239-240 of Macdonald's book on symmetric functions. Make sure your exposition includes definitions of SSYTs, weight and charge and careful, readable, and easily followable exposition of your checks. Then give an exposition of the tables on pages 350-361, including their definitions, how they were computed, what can be noticed from them, and the expectations they generate. You will be marked 40% on the quality and readability of your mathematical writing, 30% on your presentation, delivery and formatting, 20% on thoroughness and 5% on whether the answers are correct.

Assignment 4: Give a 30 min talk on one of the following topics. You will be marked 50% on the preparation, clarity, organization, thoroughness and thoughtfulness of your handwritten notes for the talk and 50% on your delivery, boardwork, clarity, coherence, organization, elegance, audience engagement, audience learning, and audience inspiration from your live lecture.

1. Flag varieties
2. Schubert polynomials
3. Macdonald polynomials
4. R-matrices and vertex models
5. Free probability
6. The Bruhat order
7. Chevalley groups
8. Hecke algebras
9. Reflection groups
10. q-t-Catalan
11. q-t-Kostka
12. Card shuffling
13. Farahat-Higman
14. Brauer algebras
15. Partition algebras
16. Kronecker products
17. Plethysm
18. Schur-Weyl duality

19. Symmetric group representations and Murphy elements
20. Affine crystals
21. Hall algebras
22. Hypergeometric functions
23. Matroids
24. Nilpotent orbits
25. Hall-Littlewood polynomials and spherical functions
26. Littlewood-Richardson coefficients
27. Jack polynomials
28. Moment graphs
29. Kazhdan-Lusztig polynomials
30. The inifinite symmetric group
31. The affine Weyl group
32. Poset Laplacians
33. Stanley-Reisner rings
34. Chromatic quasisymmetric symmetric functions
35. polytopes, zonotopes, h-vectors and f-vectors
36. Loop erased walks
37. Combinatorics of the free Lie algebra and the partition algebra

2 In preparation Exam problem list

1. Prove that
$$(x+y)^k = \sum_{r=0}^{k} \binom{k}{r} x^r y^{k-r}.$$

2. Prove that
$$(x_1+\cdots+x_n)^k = \sum_{\lambda\in Y_n} f_\lambda s_\lambda.$$

3. Prove that
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n}{k-1}.$$

4. Let f_λ be the number of standard tableaux of shape λ. Prove that
$$f_\lambda = \sum_{\substack{\mu\subseteq\lambda \\ \lambda/\mu=\square}} f_\mu.$$

5. Prove that if $xy=yx$ then $e^{x+y}=e^x e^y$.

6. Prove that
$$\log(1+z) = \sum_{k\in\mathbb{Z}_{>0}} (-1)^{k-1}\frac{1}{k}z^k \qquad \text{and} \qquad -\log(1-z) = \sum_{k\in\mathbb{Z}_{>0}} \frac{1}{k}z^k.$$

7. Prove that

$$\exp\Big(\sum_{k\in\mathbb{Z}_{>0}} \frac{u^k}{k}\Big) = \frac{1}{1-u} \qquad \text{and} \qquad \exp\Big(\sum_{k\in\mathbb{Z}_{>0}} p_k(x)\frac{u^k}{k}\Big) = \prod_{i=1}^{n} \frac{1}{1-x_iu}.$$

8. Define Young's lattice and prove that it is a lattice.

9. Let Y be Young's lattice and let $\lambda \in Y$. Give a bijection from the set of paths from $\emptyset$ to λ to the set of standard tableaux of shape λ.

10. Let f_λ be the number of standard tableaux of shape λ. Prove that

$$f_\lambda = \frac{n!}{\prod_{b\in\lambda}(a(b)+l(b)+1)}, \qquad \text{where } a(b) = \#\mathrm{arm}_\lambda(b) \text{ and } l(b) = \#\mathrm{arm}_\lambda(b).$$

11. (Vadermonde determinant) Prove that

$$\det\begin{pmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \prod_{1\le i<j\le n} (x_i - x_j).$$

12. (t-Vandermonde) Prove that

$$\sum_{w\in S_n} T_w x^\rho = \prod_{1\le i<j\le n} (x_j - tx_i).$$

13. (Wronski's relations) Prove that

$$\sum_{i+j=k} (-1)^i e_i h_j = 0.$$

14. (q-t Wronski's relations) Prove that

$$\sum_{i+j=k} (-1)^i (t^i q^j - 1) e_i g_j = 0.$$

15. (a) Define the Koszul complex for $\mathbb{C}^n$.
    (b) Show that the Koszul complex is a complex of $GL_n(\mathbb{C})$ modules.
    (c) Prove that the Koszul complex is exact.
    (d) Compute the Euler characteristic of the Koszul complex.

16. Prove that

$$h_k = \sum_{n(\mu)+|\mu|=k} (-1)^{k=|\mu|} \binom{|\mu|}{\mu} e_\mu.$$

17. Prove that

$$e_k = \sum_{n(\mu)+|\mu|=k} (-1)^{k=|\mu|} \binom{|\mu|}{\mu} h_\mu.$$

18. (Newton's relations). Prove that

$$p_k - e_1 p_{k-1} + \cdots + (-1)^{k-1} e_{k-1} p_1 + (-1)^k e_k = 0.$$

19. Prove that

$$p_k = \det \begin{pmatrix} e_1 & 1 & & & \\ 2e_2 & e_1 & 1 & & \\ \vdots & & & 1 & \\ (k-1)e_{k-1} & e_{k-2} & \cdots & e_1 & 1 \\ ke_k & e_{k-1} & \cdots e_2 & e_1 & \end{pmatrix}$$

20. Show that $q_r(X_n; 0, t) = (-t)^{r-1} e_r$, the elementary symmetric function.

21. Show that $q_r(X_n; q, 0) = q^{r-1} h_r$, the complete symmetric function.

22. Show that $q_r(X_n; q, q) = q^{r-1} p_r$, the power sum symmetric function.

23. Show that if $f(t)$ is polynomial in t with roots $\gamma_1, \ldots, \gamma_n$ then

$$\text{the coefficient of } t^r \text{ in } f(t) \text{ is } (-1)^r e_r(\gamma_1, \ldots, \gamma_n).$$

24. Show that if A is an $n \times n$ matrix with entries in $\mathbb{C}$ with eigenvalues $\gamma_1, \ldots, \gamma_n$ then the trace of the action of A on the rth exterior power of the vector space $\mathbb{C}^n$ is

$$\begin{aligned} &\mathrm{tr}(A, \Lambda^r(\mathbb{C}^n)) = e_r(\gamma_1, \ldots, \gamma_n), \quad \text{so that} \\ &\mathrm{tr}(A) = e_1(\gamma_1, \ldots, \gamma_n), \qquad \text{and} \qquad \det(A) = e_n(\gamma_1, \ldots, \gamma_n). \end{aligned}$$

25. Show that if A is an $n \times n$ matrix with entries in $\mathbb{C}$ with eigenvalues $\gamma_1, \ldots, \gamma_n$ then the chatacteristic polynomial of A is

$$\det(A - t\mathrm{id}) = \sum_{r=0}^{n} (-1)^r e_{n-r}(\gamma_1, \ldots, \gamma_n) t^r.$$

26. Show that

$$q_r = \sum_{1 \le i_1 \le \cdots \le i_r \le n} (q-t)^{\mathrm{Card}\{j \mid i_j < i_{j+1}\}} q^{\mathrm{Card}\{j \mid i_j = i_{j+1}\}} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

27. Show that

$$q_r = \sum_{i_1 \le i_2 \le \cdots \le i_k > i_{k+1} > \cdots > i_r} q^{k-1} (-t)^{r-k} x_{i_1} \cdots x_{i_k} x_{i_{k+1}} \cdots x_{i_r}.$$

28. Show that

$$q_r = \sum_{\lambda \vdash r} (q-t)^{\ell(\lambda)-1} q^{r-\ell(\lambda)} m_\lambda(x_1, \ldots, x_n).$$

29. For an $n \times \ell$ matrix $a = (a_{ij})$ with entries from $\mathbb{Z}_{\ge 0}$ let

$$\begin{aligned} rs(a) &= (\mu_1, \ldots, \mu_n), \\ cs(a) &= (\lambda_1, \ldots, \lambda_\ell), \end{aligned} \qquad \text{where} \qquad \mu_i = \sum_{j=1}^{\ell} a_{ij} \quad \text{and} \quad \lambda_j = \sum_{i=1}^{n} a_{ij},$$

so that $rs(a)$ and $cs(a)$ are the sequences of row sums and column sums of a, respectively. Define

$$x^a = x^{rs(a)} = \prod_{i=1}^{n}\prod_{j=1}^{\ell}(x_i)^{a_{ij}}, \qquad y^a = y^{cs(a)} = \prod_{j=1}^{\ell}\prod_{i=1}^{n}(y_j)^{a_{ij}}, \qquad \text{and}$$

$$\mathrm{wt}(a) = \frac{1}{(q-t)^{\ell(\lambda)}}\prod_{a_{ij}\neq 0}\left((q-t)q^{a_{ij}-1}\right) = q^{|\lambda|-\ell(a)}(q-t)^{\ell(a)-\ell(\lambda)},$$

where $\lambda = cs(a)$, $\ell(a)$ is the number of nonzero entries in a, $\ell(\lambda)$ is the number of nonzero entries in λ, and $|\lambda|$ is the sum of the entries of λ. For a sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ define

$$q_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_\ell}.$$

For a sequence $\mu = (\mu_1, \ldots, \mu_n)$ of nonnegative integers let

$$A_{\mu\lambda} = \{a \in M_{n\times\ell}(\mathbb{Z}_{\geq 0}) \mid cs(a) = \lambda,\ rs(a) = \mu\}.$$

Show that

$$q_\lambda = \sum_{\mu} a_{\mu\lambda}(q,t)m_\mu\ , \qquad \text{where} \qquad a_{\mu\lambda}(q,t) = \sum_{a\in A_{\mu\lambda}} \mathrm{wt}(a),$$

and the first sum is over partitions μ such that $|\mu| = |\lambda|$.

30. Show that

$$(t-s)q_r(X_n;t,s) + (q-t)(t-s)\left(\sum_{j=1}^{r-1} q_j(X_n;q,t)q_{r-j}(X_n;t,s)\right) + (q-t)q_r(X_n;q,t) = (q-s)q_r(X_n;q,s).$$

Use this identity to deduce that

$$q_r(X_n;q,t) + \left(\sum_{j=1}^{r-1} h_j(X_n)t^j q_{r-j}(X_n;q,t)\right) - h_r(X_n)[r]_{q,t} = 0$$

$$q_r(X_n;q,t) + \left(\sum_{j=1}^{r-1} e_j(X_n)(-q)^j q_{r-j}(X_n;q,t)\right) + (-1)^r e_r(X_n)[r]_{q,t} = 0$$

$$\sum_{j=0}^{r}(-t)^{r-j}[j]_{q,t}h_j(X_n)e_{r-j}(X_n) = (q-t)q_r(X_n;q,t).$$

and

$$rq_r(X_n;q,t) - \left(\sum_{j=1}^{r-1} p_j(X_n)(q^j-t^j)q_{r-j}(X_n;q,t)\right) - p_r(X_n)[r]_{q,t} = 0$$

and the *Newton identities*

$$kh_k = \sum_{i=1} p_i h_{k-i} \qquad \text{and} \qquad ke_k = \sum_{i=1}^{k}(-1)^{i-1}p_i e_{k-i},$$

31. Show that $\omega(q_r(X_n;q,t)) = q_r(X_n;-t,-q)$ and deduce that $\omega(p_k) = (-1)^{k-1}p_k$.

32. Prove that $\mathbb{Z}[x_1,\ldots,x_n]^{S_n} = \mathbb{Z}[e_1,\ldots,e_n]$.

33. (a) Prove that $\mathbb{Q}[x_1,\ldots,x_n]^{S_n} = \mathbb{Q}[p_1,\ldots,p_n]$.
    (b) Prove that $\mathbb{Z}[x_1,\ldots,x_n]^{S_n} \neq \mathbb{Z}[p_1,\ldots,p_n]$.

34. Prove that $\mathbb{Z}[X]^{\det} = a_\rho \mathbb{Z}[X]^{S_n}$.

35. Prove that

$$ {}_{r+1}F_r\left[\begin{matrix}\alpha_0,\alpha_1,\ldots,\alpha_r \\ \beta_1,\ldots,\beta_r\end{matrix}\ ;z\right] = \lim_{q\to 1}\left({}_{r+1}\phi_r\left[\begin{matrix}q^{\alpha_0},q^{\alpha_1},\ldots,q^{\alpha_r} \\ q^{\beta_1},\ldots,q^{\beta_r}\end{matrix}\ ;q,z\right]\right). $$

36. (q-binomial theorem). Prove that

$$ \frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(a;q)_k}{(q;q)_k} z^k. $$

37. (q-exponential function) Prove that

$$ \frac{1}{(z;q)_\infty} = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{1}{(q;q)_k} z^k \qquad \text{and} \qquad (z;q)_\infty = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(-1)^k q^{\binom{n}{2}}}{(q;q)_k} z^k. $$

38. (exponential functions) Prove that $F = \exp(z)$ and $\varphi = \exp_q(z)$ satisfy

$$ \frac{dF}{dz} = F \qquad \text{and} \qquad \varphi(qz) = \varphi(z). $$

39. (power functions) Prove that

$$ {}_1\phi_0[a;q,z] = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad \text{and} \qquad {}_1F_0[\alpha;z] = (1-z)^{-\alpha}. $$

40. (Power functions) Prove that $\varphi = {}_1\phi_0[a;q,z]$ and $F = {}_1F_0[\alpha;z] = (1-z)^{-\alpha}$ satisfy

$$ (1-zq^a)\varphi(qz) = (1-z)\varphi(z) \qquad \text{and} \qquad \frac{dF}{dz} = -\frac{a}{z}F. $$

41. (Gamma functions). Let

$$ \Gamma_q(r) = \frac{(q;q)_{r-1}}{(1-q)^{r-1}} \qquad \text{and} \qquad \Gamma(r) = r!. $$

Prove that

$$ \Gamma_q(a+1) = [a]\Gamma_q(a) \qquad \text{and} \qquad \Gamma(a+1) = a\Gamma(a). $$

Prove that

$$ \Gamma(a) = \int_0^1 e^{-t} t^{a-1}\, dt $$

Prove that

$$ \Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} \qquad \text{and} \qquad \Gamma(z) = ????. $$

42. (Beta functions) Let

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad \text{and} \qquad B(r,s) = \frac{\Gamma_q(r)\Gamma_q(s)}{\Gamma_q(r+s)}.$$

Show that

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^y \frac{dt}{(1-t)} \qquad \text{and} \qquad B(r,s) = \int_0^1 x^{r-1}(qx;q)_{s-1}\, d_q x.$$

and that the last integral is equivalent to

$$\sum_{m\in\mathbb{Z}_{\ge 0}} a^m \frac{q^{m+1};q)_\infty}{(a^{-1}bq^m;q)_\infty} = \frac{(b;q)_\infty(q;q)_\infty}{(a;q)_\infty(a^{-1}b;q)_\infty}, \qquad \text{where } a=q^r \text{ and } b=q^{r+s}.$$

43. (Gauss Hypergeometric function) Show that

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\, dt$$

and

$${}_2\varphi_1(a,b,c;;q,z) = \sum_{r\in\mathbb{Z}_{\ge 0}} \frac{(a;q)_r(b;q)_r}{(q;q)_r(c;q)_r} z^r = ???$$

Show that

$$(q^{a+b}z - q^{c-1})\varphi(q^2z) = (-(q^a+q^b)z + q^{c-1}+1)\varphi(qz) + (z-1)\varphi(z) = 0.$$

and

$$z(z-1)\frac{d^2F}{dz^2} + (c-(a+b-1)z)\frac{dF}{dz} - abF = 0.$$

44. (Weyl character formula) Show that

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} \qquad \text{and} \qquad P_\lambda(q,qt) = \frac{A_{\lambda+\rho}(q,t)}{A_\rho(q,t)}.$$

45. (Weyl denominator formula) Show that

$$a_\rho = \prod_{i<j}(x_j - x_i) \qquad \text{and} \qquad A_\rho(q,t) = \prod_{i<j}(x_j - tx_i).$$

46. (Pieri rules) Show that

$$e_r s_\lambda = ??? \qquad h_r s_\lambda = ???, \qquad p_r s_\lambda = ???, \qquad q_r s_\lambda = ???, \qquad g_r s_\lambda = ???$$

and

$$e_r P_\lambda = ??? \qquad h_r P_\lambda = ???, \qquad p_r P_\lambda = ???, \qquad q_r P_\lambda = ???, \qquad g_r P_\lambda = ???$$

47. (parabolic restriction) Show that

$$s_\lambda(x_1,\ldots,x_n) = \sum_{??} s_\mu(x_1,\ldots,x_{n-1})x_n^k \qquad \text{and} \qquad P_\lambda(x_1,\ldots,x_n) = \sum_{??} P_\mu(x_1,\ldots,x_{n-1})x_n^k$$

48. (LR rules) Show that

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \qquad \text{and} \qquad P_\mu P_\nu = \sum_\lambda c^\lambda_{\mu\nu}(q,t) P_\lambda.$$

49. (Cauchy identities) Show that

$$\prod_{i=1}^m \prod_{j=1}^n (1+x_iy_j) = \sum_{\lambda\subseteq n^m} P_\lambda(x;q,t)P_{\lambda'}(y;t;q), \qquad \prod_{i=1}^m \prod_{j=1}^n (x_i+y_j) = \sum_{\lambda\subseteq n^m} P_\lambda(x;q,t)P_{\lambda^c}(y;t;q),$$

and

$$\prod_{i=1}^m \prod_{j=1}^n \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} = \sum_{\ell(\lambda)\le \min(m,n)} b_\lambda P_\lambda(x)P_\lambda(y). = \sum_{\ell(\lambda)\le \min(m,n)} P_\lambda(x)Q_\lambda(y).$$

Then set $q=t$ to deduce that

$$\prod_{i=1}^m \prod_{j=1}^n (1+x_iy_j) = \sum_\lambda s_\lambda(x)s_{\lambda'}(y). \qquad \prod_{i=1}^m \prod_{j=1}^n (x_i+y_j) = \sum_{\lambda\subseteq n^m} s_\lambda(x)s_{\lambda^c}(y),$$

and

$$\prod_{i=1}^m \prod_{j=1}^n \frac{1}{1-x_iy_j} = \sum_\lambda s_\lambda(x)s_\lambda(y).$$

50. (Nonsymmetric Cauchy identity) Show that

$$\Big(\prod_{i,j=1}^n \frac{(qtx_iy_j;q)_\infty}{(qx_iy_j;q)_\infty}\Big)\Big(\prod_{1\le i\le j\le n} \frac{1-tx_iy_j}{1-x_iy_j}\Big) = \sum_{\lambda\in\mathbb{Z}^n_{\ge 0}} a_\lambda(q,t)E_\lambda(x;q,t)E_\lambda(y;q^{-1},t^{-1}).$$

and the left hand side is related to the character of the space of polynomial functions on the Iwahori subgroup.

51. (Jacobi-Trudi formulas) Prove the Jacobi-Trudi formulas

$$s_\lambda = \det(h_{\lambda_i - j}) \qquad \text{and} \qquad s_\lambda = \det(e_{\lambda'_i - j}).$$

52. Prove that

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda, \qquad e_\mu = \sum_\lambda K_{\lambda',\mu'} s_\lambda, \qquad p_\mu = \sum_\lambda \chi^\lambda_{S_k}(\mu) s_\lambda, \qquad q_\mu = \sum_\lambda \chi^\lambda_{H_k}(\mu) s_\lambda.$$

3 Lectures

3.1 Week 1

Lecture 1: Examples of lattices and Bratteli diagrams

Lecture 2: The binomial theorem and the exponential

Lecture 3: The symmetric group

3.1.1 Lecture 1: Examples of Lattices and Bratelli diagrams

The Young lattice

1. Define the Young lattice $\mathbb{Y}$.

2. Let $\mathcal{F}(\mathbb{Y}_{\emptyset,\lambda]}$ be the set of maximal chains from $\emptyset$ to λ and let $f_\lambda = \mathrm{Card}(\mathcal{F}(\mathbb{Y}_{\emptyset,\lambda]}$. For λ with ≤ 5 boxes compute f_λ.

3. Draw the Hasse diagram of the first 5 levels of $\mathbb{Y}$ and label each vertex with f_λ.

4. For $k \in \{1, \dots 5\}$ compute
$$\sum_{\lambda\in\mathbb{Y}_k} f_\lambda^2 \qquad \text{and} \qquad \sum_{\lambda\in\mathbb{Y}_k} f_\lambda.$$

5. Define standard Young tableau of shape λ.

6. Let $\lambda \in \mathbb{Y}_k$. Give a bijection between $\mathcal{F}(\mathbb{Y}_{[\emptyset,\lambda]})$ and the set of standard tableaux of shape λ.

The Bratelli diagram for the Brauer algebras

1. Define the Bratelli diagram for the Brauer algebras $\mathbb{B}$.

2. Let $\mathcal{F}(\mathbb{B}_{\emptyset,\lambda]}$ be the set of maximal chains from $\emptyset$ to λ and let $b_\lambda = \mathrm{Card}(\mathcal{F}(\mathbb{B}_{\emptyset,\lambda]}$.
For λ with ≤ 5 boxes compute b_λ.

3. Draw the Hasse diagram of the first 5 levels of $\mathbb{B}$ and label each vertex with b_λ.

4. For $k \in \{1, \dots 5\}$ compute
$$\sum_{\lambda\in\mathbb{B}_k} b_\lambda^2 \qquad \text{and} \qquad \sum_{\lambda\in\mathbb{B}_k} b_\lambda.$$

The Bratelli diagram for the Temperley-Lieb algebras

1. Define the Bratelli diagram for the Temperley-Lieb algebras $\mathbb{TL}$ as a sublattice of Young's lattice $\mathbb{Y}$.

2. Let $\mathcal{F}(\mathbb{TL}_{[\emptyset,\lambda]})$ be the set of maximal chains from $\emptyset$ to λ and let $f_\lambda = \mathrm{Card}(\mathcal{F}(\mathbb{TL}_{[\emptyset,\lambda]}))$.
For λ with ≤ 5 boxes compute f_λ.

3. Draw the Hasse diagram of the first 5 levels of $\mathbb{TL}$ and label each vertex with f_λ.

4. For $k \in \{1, \ldots 5\}$ compute
$$\sum_{\lambda\in\mathbb{TL}_k} f_\lambda^2 \qquad \text{and} \qquad \sum_{\lambda\in\mathbb{TL}_k} f_\lambda.$$

The Pascal lattice

1. Define the Pascal lattice as a sublattice of the Young lattice $\mathbb{P}$.

2. Let $\mathcal{F}(\mathbb{P}_{\emptyset,\lambda]}$ be the set of maximal chains from $\emptyset$ to λ and let $f_\lambda = \mathrm{Card}(\mathcal{F}(\mathbb{P}_{\emptyset,\lambda]}$. For λ with ≤ 4 boxes compute f_λ.

3. Draw the Hasse diagram of the first 5 levels of $\mathbb{P}$ and label each vertex with f_λ.

4. For $k \in \{1, \ldots 5\}$ compute
$$\sum_{\lambda\in\mathbb{P}_k} f_\lambda^2 \qquad \text{and} \qquad \sum_{\lambda\in\mathbb{P}_k} f_\lambda.$$

Standard tableaux

1. Define partition, box, $\ell(\lambda)$, $|\lambda|$, $\lambda \subseteq \mu$ and λ'. Illustrate these definitions with pictures.

2. Define arm, leg, hook length, and content of a box. Illustrate these definitions with pictures.

3. Define standard Young tableau. Illustrate this definition with illuminating pictorial examples.

4. Prove the following theorem.

Theorem 3.1. *Let $n \in \mathbb{Z}_{>0}$. Let $\lambda \in \mathbb{Y}_n$ and let f_λ be the number of standard tableaux of shape λ. For a box $b \in \lambda$ let $h_\lambda(b)$ be the hook length at the box b. Then*
$$n! = \sum_{\lambda\in\mathbb{Y}_k} f_\lambda^2 \qquad \textit{and} \qquad f_\lambda = \frac{n!}{\prod_{b\in\lambda} h_\lambda(b)}.$$

Theorem 3.2.
$$\sum_{\lambda\in\mathbb{Y}_k} f_\lambda^2 = n!, \qquad \sum_{\lambda\in\mathbb{B}_k} b_\lambda^2 = \frac{(2n)!}{2^n n!}, \qquad \sum_{\lambda\in\mathbb{TL}_k} f_\lambda^2 = \mathit{Catalan}, \qquad \sum_{\lambda\in\mathbb{P}_k} \binom{k}{\lambda}^2 = \binom{2n}{n}.$$

3.1.2 Lecture 2: The binomial theorem and the exponential

Binomial coefficents

1. Define $n!$ and $\binom{n}{k}$.
2. Calculate, by brute force, with full details suitable for a grade 8 student,
$$(x+y)^2, \quad (x+y)^3, \quad (x+y)^4, \quad \text{and} \quad (x+y)^5.$$
3. Write a careful proof of the following theorem.

   Theorem 3.3. *Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$.*

   (a) Let S be a set with cardinality n. Then $\binom{n}{k}$ *is the number of subsets of S with cardinality k.*

   (b) $\binom{n}{k}$ *is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$.*

   (c) If $k \in \{1, \ldots, n-1\}$ then
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad \textit{and} \qquad \binom{n}{0} = 1 \quad \textit{and} \quad \binom{n}{n} = 1.$$

   (d) In $\mathbb{C}[x, y]$,
$$(x+y)^n = \sum_{k=0} \binom{n}{k} x^k y^{n-k}.$$
4. Give diagrams/pictures illustrating Theorem 4.2.
5. Write a careful proof of the following proposition.

   Proposition 3.4. *For $\lambda \in \mathbb{Y}$ let f_λ be the number of standard tableaux of shape λ. If $n \in \mathbb{Z}_{>0}$ and $k \in \{1, \ldots, n\}$ then*
$$f_{(k,1^{n-k})} = \binom{n}{k}, \qquad \sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Formal power series

1. Define $\mathbb{C}((x))$, $\mathbb{C}[[x]]$ and $\mathbb{C}[x]$.
2. Carefully define $\frac{d}{dx}$.
3. Prove that $\mathbb{C}((x))$ is the field of fractions of $\mathbb{C}[[x]]$ (don't forget to prove that $\mathbb{C}[[x]]$ is an integral domain, so that you can be sure that the field of fractions is actually well defined).
4. Determine (with careful proof) $\mathbb{C}((x))^\times$, $\mathbb{C}[[x]]^\times$ and $\mathbb{C}[x]^\times$.
5. Prove that
$$\mathbb{C}((x)) = \{0\} \cup \Big(\bigsqcup_{\ell \in \mathbb{Z}} x^{-\ell} \mathbb{C}[[x]]^\times \Big).$$

6. Write a careful proof that if $\ell \in \mathbb{Z}$ then $\dfrac{d}{dx}(x^{-\ell}) = -\ell x^{-\ell-1}$.

7. Write a careful proof that if $p \in \mathbb{C}[[x]]$ then
$$p = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \text{with} \qquad a_k = \frac{1}{k!}\Big(\frac{d^k p}{dx}\Big]_{x=0}\Big).$$

8. Let $D = \frac{d}{dx}$. Let $p \in \mathbb{C}[[x]]$ and let $a \in \mathbb{C}$. Write a careful proof that
$$e^{aD}p(x) = p(x+a),$$

The exponential

1. Define the exponential.

2. Write a careful proof of the following theorem.

   Theorem 3.5.

   (a) If $xy = yx$ then $\exp(x+y) = \exp(x)\exp(y)$.

   (b) $\dfrac{d}{dx}(\exp(x)) = \exp(x)$.

3. Write a careful proof of the following theorem.

   Theorem 3.6.

   (a) If $p \in \mathbb{C}[[x]]$ and $p(x+y) = p(x)p(y)$ then
   $$\textit{there exists } a \in \mathbb{C} \quad \textit{such that} \quad p(x) = \exp(ax).$$

   (b) If $p \in \mathbb{C}[[x]]$ and $\dfrac{d}{dx}(p) = p$ then
   $$\textit{there exists } c_0 \in \mathbb{C} \quad \textit{such that} \quad p(x) = c_0\exp(x).$$

The binomial theorem

1. Define $(a)_k$ and $(a;q)_k$ and $(a;q)_\infty$.

2. Define ${}_{r+1}\phi_r$ and ${}_{r+1}F_r$.

3. Write a careful proof of the following theorem.

   Theorem 3.7. *Let $\alpha \in \mathbb{C}$. Then*
   $$(1-z)^{-\alpha} = \sum_{k\in\mathbb{Z}_{\ge 0}} \frac{(\alpha)_k}{k!} z^k = \sum_{k\in\mathbb{Z}_{\ge 0}} \binom{-\alpha}{k}(-z)^k = {}_1F_0[\alpha; z].$$

3.1.3 Lecture 3: The Symmetric group S_n

1. Carefully define the algebra of $n \times n$ matrices $M_n(\mathbb{C})$.

2. Define the matrix unit basis of $M_n(\mathbb{C})$.

3. Carefully define the general linear group $GL_n(\mathbb{C})$.

4. Define permutation, the symmetric group and describe different ways of representing a permutation.

5. Carefully define group homomorphisms $GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \to GL_{n+m}(\mathbb{C})$ and $S_n \times S_m \to S_{n+m}$ given by direct sum and product respectively. Check that these homomorphisms are well defined and injective.

6. Carefully define transpositions and simple transpositions, and the favourite element γ_μ of cycle type μ.

7. Carefully define Coxeter elements.

8. Write a careful proof of the following theorem.

   Theorem 3.8. *Let $n \in \mathbb{Z}_{>0}$.*

   (a) The function

$$\begin{array}{ccc} \{\textit{partitions of } n\} & \longrightarrow & \{\textit{conjugacy classes of } S_n\} \\ \mu & \longmapsto & [\gamma_\mu] \end{array} \qquad \textit{is a bijection.}$$

   (b) For $\mu \in \mathbb{Y}_n$ define $z_\mu = 1^{m_1} 2^{m_2} \cdots m_1! m_2! \cdots$ where m_i is the number of parts of size i in μ. Then

$$\mathrm{Card}([\gamma_\mu]) = \frac{n!}{z_\mu}.$$

3.2 Week 2: Posets and maximal chains

Lecture 4: Posets, lattices and modular lattices

Lecture 5: Maximal chains in $\mathbb{S}(n)$

Lecture 6: Maximal chains in $\mathbb{G}(\mathbb{F}_q^n)$

3.2.1 Lecture 4: Posets, lattices and modular lattices

1. Carefully define the following terms: relation, poset, Hasse diagram, supremum, infimum, lattice and modular lattice.

2. Give some favourite examples of posets and lattices.

3. Determine all posets with 3 elements and all posets with 4 elements.

4. Give three sensible precise definitions of ranked poset. Give examples to show that these are inequivalent. Discuss (with proof) conditions under which these definitions become equivalent.

5. Carefully define maximal chains.

6. Prove the following theorem.

   Theorem 3.9. *Let A be a ring and let V be an A-module. Let*

   $$\mathbb{G}(V) = \{A\text{-submodules of } V\} \qquad \text{partially ordered by inclusion.}$$

   Then $\mathbb{G}(V)$ is a modular lattice.

7. Let A be a ring and let V be an A-module. Let M and N be A-submodules of V, Show that, in $\mathbb{G}(V)$,

   $$\sup(M,N) = M+N \qquad \text{and} \qquad \inf(M,N) = M\cap N,$$

   where $M+N = \{m+n \mid m\in M, n\in N\}$

3.2.2 Lecture 5: Maximal chains in $\mathbb{G}(\mathbb{F}_q^n)$

1. Carefully define the lattice of $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$.

2. Show that $\mathbb{G}(\mathbb{F}_q^n)$ is a ranked lattice and compute $\mathrm{Card}(\mathbb{G}(\mathbb{F}_q^n))$ and $\mathrm{Card}(\mathbb{G}(\mathbb{F}_q^n)_k)$.

3. Determine the rank generating function for $\mathbb{G}(\mathbb{F}_q^n)$.

4. Prove that $\mathrm{Aut}(\mathbb{G}(\mathbb{F}_q^n)) = GL_n(\mathbb{F}_q)$.

5. Let $G = GL_n(\mathbb{F}_q)$. Define the standard flag $E = (\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_n)$ and expliciitly determine the subgroups P_k and B of G given by

   $$\mathrm{Stab}_G(E_k) = P_k \qquad \text{and} \qquad \mathrm{Stab}_G(E) = B.$$

6. Give explicit bijections

   $$G/P_k \cong \mathbb{G}(\mathbb{F}_q^n)_k \qquad G/B \cong \mathcal{F}(\mathbb{G}(\mathbb{F}_q^n)), \qquad \text{and} \qquad \mathbb{G}(\mathbb{F}_q^n)_k \cong \mathbb{G}(\mathbb{F}_q^n)_{n-k}.$$

7. Define $\mathbb{P}^1$ and $\mathbb{P}^{n-1}$ and show that
$$\mathrm{Card}(\mathbb{P}^1) = 1+q \qquad \text{and} \qquad \mathrm{Card}(\mathbb{P}^{n-1}) = 1+q+q^2+\cdots+q^{n-1}.$$

8. Let $\mathbb{C}[G/B]$ be the $\mathbb{C}$-vector space with basis indexed by the maximal chains in $\mathbb{G}(\mathbb{F}_q^n)$. For $i \in \{1,\ldots,n-1\}$ define $s_i\colon \mathbb{C}[G/B] \to \mathbb{C}[G/B]$ given by
$$T_i(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{V_{i-1}\subsetneq W \subsetneq V_{i+1}} (\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_{i+1} \subsetneq \cdots \subsetneq V_n)$$
Prove that if $i,j \in \{1,\ldots,n-1\}$ with $j \notin \{i-1,i+1\}$ and $k \in \{1,\ldots,n-2\}$ then
$$T_i^2 = (q-1)T_i + q, \quad T_iT_j = T_jT_i, \qquad T_kT_{k+1}T_k = T_{k+1}T_kT_{k+1}$$
and if $g \in \mathrm{Aut}(S_n)$ and $i \in \{1,\ldots,n\}$ then
$$gT_i = T_ig, \quad \text{as operators on } \mathbb{C}S_n.$$

3.2.3 Lecture 6: Maximal chains in $\mathbb{S}(n)$

HW questions Lecture 6

1. Carefully define the lattice of subsets of $\{1,\ldots,n\}$.
2. Show that $\mathbb{S}(n)$ is a ranked lattice and compute $\mathrm{Card}(\mathbb{S}(n)$ and $\mathrm{Card}(\mathbb{S}(n)_k)$.
3. Determine the rank generating function for $\mathbb{S}(n)$.
4. Prove that $\mathrm{Aut}(\mathbb{S}(n)) = S_n$.
5. Give a bijection between $\mathcal{F}(\mathbb{S}(n))$ and S_n.
6. Let $\mathbb{C}\mathcal{F}(\mathbb{S}(n))$ be the $\mathbb{C}$-vector space with basis indexed by the maximal chains in $\mathbb{S}(n)$. For $i \in \{1,\ldots,n-1\}$ define $s_i\colon \mathbb{C}S_n \to \mathbb{C}S_n$ given by
$$s_i(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{V_{i-1}\subsetneq W \subsetneq V_{i+1}} (\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_{i+1} \subsetneq \cdots \subsetneq V_n)$$
Prove that if $i,j \in \{1,\ldots,n-1\}$ with $j \notin \{i-1,i+1\}$ and $k \in \{1,\ldots,n-2\}$ then
$$s_i^2 = 1, \quad s_is_j = s_js_i, \qquad s_ks_{k+1}s_k = s_{k+1}s_ks_{k+1}$$
and if $g \in \mathrm{Aut}(S_n)$ and $i \in \{1,\ldots,n\}$ then
$$gs_i = s_ig, \quad \text{as operators on } \mathbb{C}S_n.$$
7. Prove that the symmetric group S_n is presented by generators $s_1,\ldots,s_n$ and relations
$$s_i^2 = 1, \quad s_is_j = s_js_i, \qquad s_ks_{k+1}s_k = s_{k+1}s_ks_{k+1}$$
for $i,j \in \{1,\ldots,n-1\}$ with $j \notin \{i-1,i+1\}$ and $k \in \{1,\ldots,n-2\}$.

3.3 Week 3: Fundamental symmetric functions

3.3.1 Lecture 7: The pheqg functions

1. Define the Pochhammer symbols
2. Define g_r, q_r, h_r, and e_r
3. State the relation between the g_r, q_r h_r and e_r and Macdonald polyomials, Hall-Littlewood polynomials and Schur functions.
4. Define the extended functions $\tilde{g}_r$ and $\tilde{q}_r$ and explain why the extend functions are not really extended.
5. Explain how $\tilde{q}_r$, q_r, h_r and e_r are specializations of $\tilde{g}_r$.
6. Derive formulas for $\tilde{q}_r$, q_r, h_r and e_r in terms of sequences.
7. define monomial symmetric functions and give some examples
8. State the q-binomial theorem
9. Find the expansions of $\tilde{g}_r$, $\tilde{q}_r$, g_r q_r, h_r and e_r in terms of monomial symmetric functions

3.3.2 Lecture 8: The power sum symmetric functions

1. Define the power sum symmetric functions
2. Derive the power sum expansion of the single Cauchy-Macdonald kernel
3. Fine the power sum expansions of $\tilde{g}_r$, $\tilde{q}_r$, g_r, q_r, h_r and e_r.
4. Derive the monomial and power sum expansion of the full Cauchy-Macdonald kernel
5. Find the monomial expansions of $\tilde{g}_\lambda$, $\tilde{q}_\lambda$, g_λ, q_λ, h_λ and e_λ in terms of matrices with specified row and column sum

3.3.3 Lecture 9: Binomial theorems

1. Derive the binomial theorems for $(1+z)^n$ and $(1+z)^{-n}$,
2. Derive the finie q-binomial theorems
3. Derive the prinicipal specializations of g_r, q_r, h_r and e_r.
4. State and prove the infiinite q-binomial theorem
5. Establish the important specializations of the infinite q-binomial theorem

3.3.4 Lecture 9: Wronski identities, Jacobi-Trudi and Giambelli formulas

1. Newton identities
2. Wronski identitiies
3. BGG resolutions
4. Koszul complex

3.4 Week 4: Crystals and Schur functions

3.4.1 Lecture 10: Crystals

1. Define the crystal $B(\square)$.
2. Define the direct sum $B_1 \oplus B_2$ of crystals.
3. Define the tensor product $B_1 \otimes B_2$ of crystals.
4. Explicitly compute $B(\square)^{\otimes 2}$ and $B(\square)^{\otimes 3}$.
5. Define the crystal of words.
6. Define crystal, morphism of crystals, isomorphism of crystals and crystal graph.

3.4.2 Lecture 11: Words and SSYTs

1. Define partitions and SSYTs.
2. Defne $B(\lambda)$ for a partition λ.
3. Prove the following theorem.

   Theorem 3.10. *There is a unique crystal structure on $B(\lambda)$ such that the arabic reading map*

   $$B_n(\lambda) \hookrightarrow B(\square)^{\otimes k} \qquad \textit{is a crystal morphism.}$$

4. Explicitly describe the crystal structure on $B(\lambda)$.
5. Explicitly construct the crystals $B(\lambda)$ for $|\lambda| \in \{1,2,3,4\}$ and $n \in \{2,3\}$.
6. Show that $\mathrm{char}(B(k)) = h_k$.
7. Show that $\mathrm{char}(B((1^k)) = e_k$.
8. Carefully define $B(\lambda)_\mu$ and $K_{\lambda\mu}$ and prove that $K_{\lambda\lambda} = 1$.
9. Carefully define the dominance order and prove that if $K_{\lambda\mu} \neq 0$ then $\mu \leq \lambda$.
10. Define x^μ and prove that

    $$\mathrm{char}(B(\lambda)) = \sum_\mu K_{\lambda\mu} x^\mu.$$

    Carefully specify what set the sum is over.
11. Define the Schur function s_λ.

3.4.3 Lecture 12: The Weyl character formula

1. Carefully define symmetric functions and the character of a crystal.

2. Carefully define i-string.

3. Let B be a crystal. For $i \in \{1, \ldots, n-1\}$ define $s_i \colon B \to B$ by
$$s_i(b) = \tilde{e}^k t, \qquad \text{where} \qquad h = \tilde{e}^k b, \tilde{e}^{k-1} b, \ldots, \tilde{e} b, b, \tilde{f} b, \ldots, \tilde{f}^\ell b = t,$$
with $\tilde{f}^\ell b \neq 0$, $\tilde{f}^{\ell+1} b = 0$, $\tilde{e}^k b \neq 0$ and $\tilde{e}^{k+1} b = 0$.
Show that the operators $s_1, \ldots, s_{n-1}$ define an S_n action on B.

4. Let B be a crystal. Show that $\mathrm{char}(B)$ is a symmetric function.

5. Define s_λ and show that s_λ is a symmetric function.

6. Define ρ and $a_{\lambda+\rho}$.

7. Let B be a crystal and let $B^+ = \{p \in B \mid \text{if } i \in \{1, \ldots, n-1\} \text{ then } \tilde{e}_i p = 0\}$. Show that
$$a_\rho \cdot \mathrm{char}(B) = \sum_{p \in B^+} a_{\mathrm{wt}(p)+\rho}.$$

8. Show that
$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}.$$

3.5 Week 5: Symmetric functions, crystals and RSK

3.5.1 Lecture 13: The Littlewood-Richardson rule

1. Let B be a crystal and let $B^+ = \{p \in B \mid \text{if } i \in \{1, \ldots, n-1\} \text{ then } \tilde{e}_i p = 0\}$. Show that
$$a_\rho \cdot \mathrm{char}(B) = \sum_{p \in B^+} a_{\mathrm{wt}(p)+\rho}.$$

2. Define irreducible crystal.

3. Prove that if B is an irreducible crystal then there is a unique $p \in B$ such that if $i \in \{1, \ldots, n-1\}$ then $\tilde{e}_i p = 0$.

4. Prove that if B is an irreducible crystal and $p \in B$ such that if $i \in \{1, \ldots, n-1\}$ then $\tilde{e}_i p = 0$ then $\mathrm{wt}(p)$ is a partition.

5. Prove that if B is an irreducible crystal and $p \in B$ is such that if $i \in \{1, \ldots, n-1\}$ then $\tilde{e}_i p = 0$ then $B \cong B(\lambda)$ where $\lambda = \mathrm{wt}(p)$.

6. Let B_1 and B_2 be crystals. Prove that
$$\mathrm{char}(B_1 \otimes B_2) = \mathrm{char}(B_1)\mathrm{char}(B_2).$$

7. Let λ, μ be partitions and let $n > \ell(\lambda) + \ell(\mu)$. Show that

$$s_\mu s_\nu = \sum_\nu \mathrm{Card}((B(\mu) \otimes B(\nu))^+_\lambda) s_\lambda,$$

where

$$(B(\mu) \otimes B(\nu))^+_\lambda = \{p \in B(\mu) \otimes B(\nu) \mid \mathrm{wt}(p) = \lambda \text{ and if } i \in \{1, \dots, n-1\} \text{ then } \tilde{e}_i p = 0\}$$

8. Define skew shape and LR fillings.
9. Give a bijection

$$(B(\mu) \otimes B(\nu))^+_\lambda \longleftrightarrow \{\text{LR fillings of shape } \lambda/\mu \text{ and weight } \nu\}$$

10. Prove the following theorem.

Theorem 3.11. *Let $c^\lambda_{\mu\nu}$ be the number of LR fillings of λ/μ of weight ν. Then*

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

3.5.2 Lecture 14: The combinatorial R-matrix and RSK

1. Explicitly describe the crystal $B(1^k)$.
2. Give a crystal isomorphism $B(\Box) \otimes B(1^k) \cong B(1^k) \otimes B(\Box)$.
3. Carefully define the RSK algorithm.
4. Use RSK to prove the following theorem.
5. Use RSK to prove the following theorem.

Theorem 3.12.

$$S_k \leftrightarrow \bigsqcup_{\lambda \in \mathbb{Y}_k} \hat{S}^\lambda_k \times \hat{S}^\lambda_k$$

$$B^{\otimes k} \leftrightarrow \bigsqcup_{\substack{\lambda \in \mathbb{Y}_k \\ \ell(\lambda) \le n}} \hat{S}^\lambda_k \times B_n(\lambda)$$

$$M_{t\times s}(\mathbb{Z}_{\ge 0}) \leftrightarrow \bigsqcup_{\ell(\lambda) \le \min(s,t)} B_t(\lambda) \times B_s(\lambda)$$

$$M_{t\times s}(\{0,1\}) \leftrightarrow \bigsqcup_{\lambda \subseteq (s^t)} B_t(\lambda) \times B_s(\lambda^c)$$

6. Carefully define the symbols in the following theorem and prove it.

Theorem 3.13.

$$k! = \sum_{\lambda\in\mathbb{Y}_k} f_\lambda^2.$$

$$(x_1+\cdots+x_n)^k = \sum_{\lambda\in\mathbb{Y}_k} f_\lambda s_\lambda.$$

$$\prod_{i=1}^t\prod_{j=1}^s \frac{1}{1-x_iy_j} = \sum_{\ell(\lambda)\le\min(s,t)} s_\lambda(x)s_\lambda(y).$$

$$\prod_{i=1}^t\prod_{j=1}^s (1+x_iy_j) = \sum_{\lambda\in(s^t)} s_\lambda(x)s_{\lambda^c}(y).$$

$$\prod_{i=1}^t\prod_{j=1}^s (x_i+y_j) = \sum_{\lambda\in(s^t)} s_\lambda(x)s_{\lambda'}(y).$$

3.5.3 Lecture 15: Pieri rules and Murnaghan-Nakayama rules

1. Define horizontal strip, vertical strip, border strip and broken border strip.
2. Use RSK to prove the following theorem.

   Theorem 3.14.

   $$q_r s_\mu = \sum_{\lambda/\mu\ \text{bbstrip}} q^{r-\mathrm{ht}(\lambda/\mu)}(-t)^{\mathrm{ht}(\lambda/\mu)} s_\lambda,$$

   $$h_r s_\mu = \sum_{\lambda/\mu\ \text{hstrip}} s_\lambda, \qquad e_r s_\mu = \sum_{\lambda/\mu\ \text{vstrip}} s_\lambda, \qquad p_r s_\mu = \sum_{\lambda/\mu\ \text{bstrip}} (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda.$$

3. Carefully define $K_{\lambda\mu}$, $\chi^\lambda_{S_k}(\mu)$, $\chi^\lambda_{H_k}(\mu)$ and prove the following theorem.

   Theorem 3.15.

   $$q_\mu = \sum_\lambda \chi^\lambda_{H_k}(\mu) s_\lambda.$$

   $$h_\mu = \sum_\lambda K_{\lambda\mu}s_\lambda, \qquad e_\mu = \sum_\lambda K_{\lambda'\mu'}s_\lambda, \qquad p_\mu = \sum_\lambda \chi^\lambda_{S_k}(\mu)s_\lambda.$$

3.6 Week 6: Catalan combinatorics

3.6.1 Lecture 16: q-t-Catalan and Dyck paths

1. Define the Shi arrangement, parking functions, dinv and area, and the q-t-RLT polynomial
2. Define Dyck paths, dominant Shi reagions, area, bounce and the q-t-Catalan polynomial
3. Prove that $C_{d/n}(1,1) = \frac{1}{d+n}\binom{d+n}{n}$
4. Prove the recusion for Catalan $C_{d/n}(1,1)$
5. Determine the generating function for $C_{d/n}(1,1)$.

3.6.2 Lecture 17: ∇e_n and diagonal coinvariants

1. Define the DAHA module $L_{d/n}(triv)$
2. Define the parking module $Park_{d/n}$
3. Define the Garsia-Haiman module $GH_{d/n}$
4. Define $H^*(Y_{\mathfrak{b}}^{-1}(\gamma^d))$
5. Prove that
$$L_{d/n}(triv) \cong Park_{d/n} \cong GH_{d/n} \cong H^*(Y_{\mathfrak{b}}^{-1}(\gamma^d)).$$
6. Prove that
$$\mathrm{grdim}(H^*(Y_{\mathfrak{b}}^{-1}(\gamma^d))) = R_{d//n}(q,t).$$
7. Prove that
$$\mathrm{grdim}(H^*(Y_{\mathfrak{b}}^{-1}(\gamma^d))^{\det}) = C_{d//n}(q,t).$$

3.6.3 Lecture 18: Modified Macdonald polynomials and Garsia-Haiman modules

1. Define the modified Macdonald polynomial
2. Define the nabla operator
3. Prove that
$$\nabla e_n = \mathrm{grch}(H^*(Y_{\mathfrak{b}}^{-1}(\gamma^{n+1}))).$$

3.7 Week 7: $GL_n(\mathbb{F}_q)$ and G/B

3.7.1 Lecture 19: Generators and relations for $GL_n(\mathbb{F})$

HW questions

1. Prove the following theorem

   Theorem 3.16.
$$G = \bigsqcup_{w\in S_n} BwB$$
   If $w = s_{i_1}\cdots_{i_\ell}$ is a reduced word then
$$BwB = \{y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)B \mid c_1,\ldots,c_\ell\}.$$

2. Prove that
$$\mathrm{Card}(BwB/B) = q^{\ell(w)} \qquad \text{and} \qquad \mathrm{Card}(G/B) = \sum_{w\in S_n} q^{\ell(w)}.$$

3.7.2 Lecture 20: The Bruhat decomposition and the Poincaré polynomial

1. Define $x_{ij}(c)$, $d_i(c)$, s_i, s_{ij} and $y_i(c)$.
2. State and prove a presentation theorem for $GL_n(\mathbb{F})$.

3.7.3 Lecture 21: Schubert varieties and Grassmannians

1. Simple transpositions and reduced words
2. reduced words for the longest element
3. Roots, Inversions and subwords

Theorem 3.17. *The length gen function of W is*

$$\prod_{i=1}^{n} \frac{(1-t^{d_i})}{(1-t)}.$$

3.8 Week 8: Moment graphs and Kazhdan-Lusztig polynomials

3.8.1 Lecture 22: Moment graphs and $H_T(G/B)$

3.8.2 Lecture 23: Sheaves on moment graphs

3.8.3 Lecture 24: Kazhdan-Lusztig polynomials

3.9 Week 9: Springer fibers

3.9.1 Lecture 25: Cells in Springer fibers

3.9.2 Lecture 26: Modified Hall-Littlewood polynomials

3.9.3 Lecture 27: AFL Grand Final Eve Holiday

3.10 Week 10: More Catalan combinatorics

3.10.1 Lecture 25: The Temperley-Lieb algebra

3.10.2 Lecture 26: The noncrossing partition lattice

1. tranpositions and the absolute order
2. The noncrossing partition lattice
3. Murphy elements

Theorem 3.18. *The rank gen function of $\mathcal{C}_{[1,c]}$ is*

$$\prod_{i=1}^{n} 1 + (d_i - 1)q$$

$$\mathrm{Card}(\mathcal{C}_{[1,c]}) = Catalan, \qquad \#\{maximal\ chains\ in\ \mathcal{C}_{[1,e]}\} = \frac{1}{|W|}h^n n!.$$

3.10.3 Lecture 24: binary tree and rooted labeled trees

1. Determine the number of binary tress with n internal nodes
2. Determine the number of rooted labeled trees
3. Define the corresponding operads
4. Explain the Kontsevich integral

3.11 Additional material

3.11.1 Lecture 10: special functions and differential equations

1. powers, exponential functions, gamma functions, beta integrals
2. hypergometric functions

3.11.2 Lecture 11: The q-binomial theorem

1. The theorem as a specializatoin of a symmetric function identity
2. Jackson integrals

3.11.3 Lecture 12: q-hypergeometric functions

1. The difference equation
2. Selberg integrals

3.11.4 Lecture 18: Schubert polynomials

1. Definition of Schubert vareities
2. generalised cohomology
3. The Borel presentation (i.e. the coinvariant ring)
4. push-pulls, Schubert polynomials and Grothendieck polynomials
5. Schubert classes for Grassmannians
6. Monk's rule

3.11.5 Lecture 18: Reflection groups

1. Definitions, reflections, simple reflections, roots
2. Bruhat order
3. $\mathcal{C}_{[1,c]}$

4 Week 1: Partitions, binomial coefficients, symmetric group

4.1 Partitions and the Young lattice

Young's lattice (boxes in a corner)

PICTURE

Brauer Bratteli diagram (add and remove)

PICTURE

Temperley-Lieb Bratteli diagram (restrict to two rows)

PICTURE

Pascal's triangle (restrict to along the wall)

PICTURE

A *partition* is $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\ell \in \mathbb{Z}_{\geq 0}$, $\lambda_1, \ldots, \lambda_\ell$ and $\lambda_1 \geq \cdots \geq \lambda_\ell$.
A *box* is an element of $\mathbb{Z}^2$.
Identify $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with a set of boxes

$$\lambda = \}(r, c) \in \mathbb{Z} \times \mathbb{Z} \mid r \in \{1, \ldots, \ell\} \text{ and } c_r \in \{1, \ldots, \lambda_r\} \},$$

so that λ has λ_r boxes in row r.

$$\lambda = (53311) = TABLEAU = \left\{ \begin{array}{lllll} (1,1), & (1,2), & (1,3), & (1,4), & (1,5) \\ (2,1), & (2,2), & (2,3), & & \\ (3,1), & (3,2), & (3,3), & & \\ (4,1), & & & & \\ (5,1), & & & & \end{array} \right\}$$

Let

$$\ell(\lambda) = \ell \qquad \text{and} \qquad |\lambda| = \lambda_1 + \cdots \lambda_\ell,$$

if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. Write

$$\lambda \subseteq \mu \quad \text{if } \lambda \text{ is a subset of } \mu.$$

The *conjugate* of λ is

$$\lambda' = \{(c, r) \mid (r, c) \in \lambda\}.$$

For $n \in \mathbb{Z}_{\geq 0}$ let

$$\mathbb{Y}_n = \{\text{partitions } \lambda \text{ with } |\lambda| = n\} \qquad \text{and} \qquad \mathbb{Y} = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Y}_n.$$

Let $\lambda \in \mathbb{Y}_n$ and identify λ with the set of boxes of λ. A *standard tableau of shape* λ is a function $T: \lambda \to \{1, \ldots, n\}$ such that

(a) If $(r, c), (r, c+1) \in \lambda$ then $T(r, c) < T(r, c+1)$.

(b) If $(r,c),(r+1,c)\in\lambda$ then $T(r,c)<T(r+1,c)$.

Let
$$\hat S_n^\lambda = \{\text{standard tableaux of shape } \lambda\} \qquad \text{and} \qquad f_\lambda = \mathrm{Card}(\hat S_n^\lambda).$$

If $(r,c)\in\lambda$ define
$$\mathrm{arm}(r,c)=\lambda_r - r, \qquad \qquad \mathrm{leg}(r,c)=\lambda'_c - c.$$

PICTURE

Theorem 4.1. *Let $n\in\mathbb{Z}_{>0}$ and $\lambda\in\mathbb{Y}_n$. Then*
$$f_\lambda = \frac{n!}{\prod_{b\in\lambda}(\mathrm{arm}(b)+\mathrm{leg}(b)+1)} \qquad \textit{and} \qquad n! = \sum_{\lambda\in\mathbb{Y}_n} f_\lambda^2.$$

4.2 The binomial theorem and the exponential

4.2.1 Binomial coefficients

For $k\in\mathbb{Z}_{\ge 0}$ define
$$0!=1 \quad \text{and} \quad k! = k(k-1)\cdots 3\cdot 2\cdot 1, \quad \text{for } k\in\mathbb{Z}_{>0}.$$

For $k\in\{0,1,\ldots,n\}$ define
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem 4.2. *Let $n,k\in\mathbb{Z}_{\ge 0}$ with $k\le n$.*

(a) Let S be a set with cardinality n. Then
$\binom{n}{k}$ is the number of subsets of S with cardinality k.

(b) $\binom{n}{k}$ is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$.

(c) If $k\in\{1,\ldots,n-1\}$ then
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad \textit{and} \qquad \binom{n}{0}=1 \quad \textit{and} \quad \binom{n}{n}=1.$$

(d) In $\mathbb{C}[x,y]$,
$$(x+y)^n = \sum_{k=0} \binom{n}{k} x^k y^{n-k}.$$

$$\begin{matrix} &&&&\binom{0}{0}&&&& \\ &&&\binom{1}{0}&&\binom{1}{1}&&& \\ &&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&& \\ &\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}& \\ \binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4} \end{matrix} \quad = \quad \begin{matrix} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{matrix}$$

Corollary 4.3.
$$f_{(k,1^{n-k})} = \binom{n}{k}, \qquad \sum_{k=0}^{n}\binom{n}{k} = 2^n, \qquad \sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

Proof. Flath see Halverson-Herbig arxiv:0806.3960 page 3. □

4.2.2 Formal power series

The ring of *formal power series* is

$$\mathbb{C}[[x]] = \{a_0 + a_1x + a_2x^2 + \cdots \mid a_i \in \mathbb{C}\}$$

and its field of fractions is the *ring of expressions*,

$$\mathbb{C}((x)) = \{a_{-\ell}x^{-\ell} + a_{-\ell+1}x^{-\ell+1} + a_{-\ell+2}2x^{-\ell+2} + \cdots \mid \ell \in \mathbb{Z}, a_i \in \mathbb{C}\},$$

and the ring of polynomials is

$$\mathbb{C}[x] = \{a_0 + a_1x + a_2x^2 + \cdots \mid a_i \in \mathbb{C} \text{ and all but a finite number of the } a_i \text{ are } 0\}.$$

4.2.3 The exponential

The *exponential* is

$$\exp(x) = e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots.$$

This is the most important expression in mathematics.

Theorem 4.4.

(a) If $xy = yx$ then $\exp(x+y) = \exp(x)\exp(y)$.

(b) $\dfrac{d}{dx}\exp(x) = \exp(x)$.

Theorem 4.5.

(a) If $p \in \mathbb{C}[[[x]]$ and $p(x+y) = p(x)p(y)$ then

$$\textit{there exists} \quad a \in \mathbb{C} \qquad \textit{such that} \qquad p(x) = \exp(ax).$$

(b) If $p \in \mathbb{C}[[x]]$ and $\frac{dp}{dx} = p$ then

$$\textit{there exists} \quad c_0 \in \mathbb{C} \qquad \textit{such that} \qquad p(x) = c_0\exp(x).$$

4.2.4 The binomial theorem

Let

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) \qquad \text{and} \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1).$$

Define

$${}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix} ; q, z\right] = \sum_{k\in\mathbb{Z}_{\ge 0}} \frac{(a_0;q)_k(a_1;q)_k\cdots(a_r;q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k} z^k.$$

and

$${}_{r+1}F_r\left[\begin{matrix} \alpha_0, \alpha_1, \ldots, \alpha_r \\ \beta_1, \ldots, \beta_r \end{matrix} ; z\right] = \sum_{k\in\mathbb{Z}_{\ge 0}} \frac{(\alpha_0)_k(\alpha_1)_k\cdots(\alpha_r;q)_k}{(1)_k(\beta_1)_k\cdots(\beta_r)_k} z^k,$$

If $\alpha \in \mathbb{Z}_{>0}$ then

$$(\alpha)k = \frac{(\alpha+k-1)!}{(\alpha-1)!} \qquad \text{so that} \quad n! = (1)_n \quad \text{and} \quad \binom{n}{k} = \frac{(k)_{n-k}}{(1)_k}$$

when $n, k \in \mathbb{Z}_{>0}$ with $k \le n$.

Theorem 4.6. *Let $\alpha \in \mathbb{C}$. Then*

$$(1-z)^{-\alpha} = \sum_{k\in\mathbb{Z}_{\ge 0}} \frac{(\alpha)_k}{k!} z^k = \sum_{k\in\mathbb{Z}_{\ge 0}} \binom{-\alpha}{k} (-z)^k = {}_1F_0[\alpha; z]$$

Proof. (One option) Taylor series:

$$\frac{1}{k!}\frac{d^k}{dx^k}(1+x)^\alpha\Big]_{x=0} = \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k!}.$$

□

4.3 The symmetric group

Let $n \in \mathbb{Z}_{>0}$. The vector space of $n \times n$ matrices

$$M_n(\mathbb{C}) \quad \text{has } \mathbb{C}\text{-basis} \quad \{E_{ij} \mid i,j \in 1,\dots,n\},$$

whee E_{ij} is the matrix with 1 in the (i,j) entry and 0 elsewhere.
A *permutation of* n is $w \in M_{n\times n}(\mathbb{C})$ such that

(a) There is exactly one nonzero entry in each row and each column.
(b) The nonzero entries are 1.

The *symmetric group* is the set

$$S_n = \{w \in M_{n\times n}(\mathbb{C}) \mid w \text{ is a permutation of } \{1,\dots,n\}\}$$

with matrix multiplication. Identify a permutation $w \in M_{n\times n}(\mathbb{C})$ with a bijection $w\colon \{1,\dots,n\} \to \{1,\dots,n\}$ by

$$w(i) = j \qquad \text{if } w_{ji} = 1,$$

where w_{ij} is the (i,j)-entry of the matrix w. The *transpositions*, or *reflections*, in S_n are

$$s_{ij} = 1 + E_{ij} + E_{ji} - E_{ii} - E_{jj}, \qquad \text{for } i,j \in \{1,\dots,n\} \text{ with } i \ne j.$$

The *simple transositions* are

$$s_1 = s_{12}, \qquad s_2 = s_{23}, \quad \dots, \quad s_{n-1} = s_{n-1,n}.$$

The *general linear group* is the set

$$GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \text{there exists } A^{-1} \in M_n(\mathbb{C}) \text{ with } AA^{-1} = 1 \text{ and } A^{-1}A = 1\}$$

with matrix multiplication.

Proposition 4.7. *The maps*

$$\begin{array}{ccc} GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) & \longrightarrow & GL_{n+m}(\mathbb{C}) \\ (A,B) & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S_n \times S_m & \longrightarrow & S_{n\times m} \\ (v,w) & \longmapsto & \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \end{array}$$

are injective group homomorphisms.

Let $|gamma_1 = E_{11}$ in S_1 and

$$\gamma_k = E_{12} + E_{23} + \cdots + E_{k-1,k} + E_{k1} \quad \text{in } S_k,$$

for $k \in \mathbb{Z}_{>1}$. For $\mu_1, \ldots, \mu_\ell \in \mathbb{Z}_{>0}$ let

$$\gamma_\mu = \gamma_{\mu_1} \times \cdots \times \gamma_{\mu_\ell} \quad \text{in} \quad S_{\mu_1} \times \cdots S_{\mu_\ell} \subseteq S_{\mu_1+\cdots+\mu_\ell}.$$

A *Coxeter element* of S_n is an element of the conjugacy class of γ_n in S_n. For $\mu_1, \ldots, \mu_\ell \in \mathbb{Z}_{>0}$ let $n = \mu_1 + \ldots + \mu_\ell$ and let

$$[\gamma_\mu] \quad \text{denote the conjugacy class of } \gamma_\mu \text{ in } S_n.$$

A *partition of* n is $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that $\lambda_1, \ldots, \lambda_\ell \in \mathbb{Z}_{>0}$ and $\lambda_1 \geq \cdots \geq \lambda_\ell$ and $\lambda_1 + \cdots + \lambda_\ell = n$.

Theorem 4.8.

(a) The map

$$\begin{matrix} \{\textit{partitions of } n\} & \longrightarrow & \{\textit{conjugacy classes of } S_n\} \\ \lambda & \longmapsto & [\gamma_\lambda] \end{matrix} \qquad \textit{is a bijection.}$$

(b) If λ is a partition of n and m_i is the number of parts of size i (write $\lambda = (1^{m_1} 2^{m_2} \cdots)$) then

$$\mathrm{Card}([\gamma_\lambda]) = \frac{n!}{z_\lambda}, \qquad \textit{where} \quad z_\lambda = (1^{m_1} 2^{m_2} \cdots)(m_1! m_2! \cdots).$$

Proof. For example, if $w = (531624)$ then

$$PICTURE = PICTURE = \gamma_{42}$$

and if

$$\gamma_\lambda = \gamma_1 \times \gamma_1 \times \gamma_1 \times \gamma_1 \times \gamma_2 \times \gamma_2 \times \gamma_2 \times \gamma_3 \times \gamma_4 \times \gamma_4$$

then

$$\begin{aligned} \mathrm{Card}(\mathrm{Stab}(\gamma_\lambda)) &= 4! \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 3! \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2! \cdot 4 \cdot 4 \\ &= 4! \cdot 1^4 \cdot 3! \cdot 2^3 \cdot 1! \cdot 3^1 \cdot 2! \cdot 4^2 \end{aligned}$$

so that

$$\mathrm{Card}([\gamma_\lambda]) = \frac{\mathrm{Card}(S_n)}{\mathrm{Card}(\mathrm{Stab}(\gamma_\lambda))} = \frac{n!}{z_\lambda}.$$

□

5 Weeks 2: Posets and maximal chains

5.1 Posets and lattices

Two exaxmples:

(1) Let $n \in \mathbb{Z}_{>0}$. The *subset lattice of* $\{1, \ldots, n\}$ is

$$\mathbb{S}(n) = \{\text{subsets of } \{1, \ldots, n\}\} \qquad \text{partially ordered by inclusion.}$$

(2) The *Young lattice*is

$$\mathbb{Y} = \{\text{partitions}\} \qquad \text{partially ordered by inclusion.}$$

PICTURE *PICTURE*
The subset lattice $\mathbb{S}(3)$ The Young lattice $\mathbb{Y}$

Then

$$\mathbb{S}(n) = \bigsqcup_{k=0}^{n} \mathbb{S}(n)_k, \qquad \text{where} \quad \mathbb{S}(n) = \{\text{subsets of } \{1, \ldots, n\} \text{ with cardinality } k\},$$

and

$$\mathbb{Y} = \bigsqcup_{n=0}^{\infty} \mathbb{Y}_n, \qquad \text{where} \quad \mathbb{Y}_n = \{\text{partitions with } n \text{ boxes}\}.$$

5.1.1 Posets

Let S be a set. A *relation on* S is a subset of $S \times S$.

$$\text{Write} \qquad x \le y \qquad \text{if } (x, y) \text{ is in the relation } \le.$$

A *partially ordered set*, or *poset*, is a set P with a relation $\le$ on P such that

(a) If $x \in P$ then $x \le x$,

(b) If $x, y, z \in P$ and $x \le y$ and $y \le z$ then $x \le z$, and

(c) If $x, y \in P$ and $x \le y$ and $y \le x$ then $x = y$.

The *Hasse diagram* of P is the graph with

$$\text{Vertices:} P \qquad \text{and} \qquad \text{Directed edges: } x \to y \text{ if } x \le y.$$

A *maximal chain in* P is a function

$$\begin{matrix} \mathbb{Z}_{>0} & \to & P \\ i & \mapsto & x_i \end{matrix} \quad \text{such that} \quad \begin{matrix} \text{(a) if } i \in \mathbb{Z}_{>0} \text{ then } x_i < x_{i+1}, \\ \text{(b) There does not exist } y \in P \text{ such that } x_i < y < x_{i+1}. \end{matrix}$$

5.1.2 Lattices

Let P be a poset and $E \subseteq P$.

The *infiumum*, or *greatest lower bound of* E *in* P is an element $\ell \in P$ such that

(a) If $p \in E$ then $\ell \le p$,

(b) If $m \in P$ and m satisfies the condition (if $p \in E$ then $m \le p$) then $m \le \ell$.

The *supremum*, or *least upper bound of* E *in* P is an element $\gamma \in P$ such that

(a) If $p \in E$ then $p \ge \gamma$,

(b) If $\tau \in P$ and τ satisfies the condition

$$\text{if } p \in E \text{ then } p \le \tau$$

then $\gamma \le \tau$.

For $k \in \mathbb{Z}_{>0}$ and $x_1, \ldots, x_k \in P$ use the notation

$$\inf(x_1, \ldots, x_k) = \inf(\{x_1, \ldots, x_k\}) \qquad \text{and} \qquad \sup(x_1, \ldots, x_k) = \sup(\{x_1, \ldots, x_k\}).$$

A *lattice* is a poset P such that

$$\text{if } x, y \in P \text{ then } \inf(x, y) \text{ and } \sup(x, y) \text{ exist in } P.$$

5.1.3 Modular lattices

Let P be a lattice. Use the notation

$$x \wedge y = \inf(x, y) \qquad \text{and} \qquad x \vee y = \sup(x, y),$$

and the language is $x \wedge y$ is "x meet y" and $x \vee y$ is "x join y".
A *modular lattice* is a lattice P such that

$$\text{if } m, n, p \in P \text{ and } p \le m \quad \text{then} \quad m \vee (n \wedge p) = (m \vee p) \wedge n.$$

Theorem 5.1. *Let A be a $\mathbb{Z}$-algebra and let V be an A-module. Let*

$$\mathbb{G}(V) = \{A\text{-submodules of } V\} \qquad \text{partially ordered by inclusion.}$$

Then $\mathbb{G}(V)$ is a modular lattice.

Proposition 5.2. *Let A be a $\mathbb{Z}$-algebra and let V be an A-module. Let $M, N, P \in \mathbb{G}(V)$.*

(a) (infimums exist)

$$\inf(M, N) = M \cap N = \{v \in V \mid v \in M \text{ and } v \in N\}.$$

(b) (supremums exist)

$$\sup(M, N) = M + N = \{m + n \mid m \in M \text{ and } n \in N\}.$$

(c) (modular law)

$$\text{If } P \subseteq M \quad \text{then} \quad M + (N \cap P) = (M+) \cap P.$$

(d) (modular property)

$$\frac{M + N}{M} \cong \frac{N}{M \cap N}.$$

Proposition 5.3. *Let $A = \mathbb{F}_q$ and let V be an A-module so that $V = \mathbb{F}_q^n$, where $n = \dim(V)$ as an $\mathbb{F}_q$-vector space. Then*

$$\mathbb{G}(\mathbb{F}_q^n) = \bigsqcup_{k=0}^{n} \mathbb{G}(\mathbb{F}_q^n)_k, \qquad \text{where} \quad \mathbb{G}(FF_q^n)_k = \{\mathbb{F}_q\text{-subspaces } W \text{ of } \mathbb{F}_q^n \text{ with } \dim(W) = k\}$$

and

$$\mathrm{Card}(\mathbb{G}(\mathbb{F}_q^n)_k) = \begin{bmatrix} n \\ k \end{bmatrix}.$$

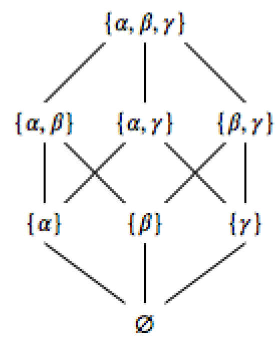
5.2 Partially ordered sets

Let S be a set.

- A *partial order* on S is a relation $\le$ on S such that
  (a) If $x \in A$ then $x \le x$,
  (b) If $x, y, z \in S$ and $x \le y$ and $y \le z$ then $x \le z$, and
  (c) If $x, y \in S$ and $x \le y$ and $y \le x$ then $x = y$.
- A *total order* on S is a partial order $\le$ such that

(d) If $x, y \in S$ then $x \le y$ or $y \le x$.

- A *partially ordered set*, or *poset*, is a set S with a partial order $\le$ on S.
- A *totally ordered set* is a set S with a total order $\le$ on S.



The poset of subsets of $\{\alpha, \beta, \gamma\}$ with inclusion as $\le$

Let S be a poset. Write

$$x < y \quad \text{if} \quad x \le y \text{ and } x \ne y.$$

- The *Hasse diagram* of S is the graph with vertices S and directed edges given by

$$x \to y \qquad \text{if } x \le y.$$

- A *lower order ideal of* S is a subset E of S such that

$$\text{if } y \in E \text{ and } x \in S \text{ and } x \le y \quad \text{then} \quad x \in E.$$

- The *intervals in* S are the sets

$$\begin{array}{ll}
S_{[a,b]} = \{x \in S \mid a \le x \le b\} & S_{(a,b)} = \{x \in S \mid a < x < b\} \\
S_{[a,b)} = \{x \in S \mid a \le x < b\} & S_{(a,b]} = \{x \in S \mid a < x \le b\} \\
S_{(-\infty,b]} = \{x \in S \mid x \le b\} & S_{[a,\infty)} = \{x \in S \mid a \le x\} \\
S_{(-\infty,b)} = \{x \in S \mid x < b\} & S_{(a,\infty)} = \{x \in S \mid a < x\}
\end{array}$$

for $a, b \in S$.

5.2.1 Upper and lower bounds, sup and inf

Let S be a poset and let E be a subset of S.

- An *upper bound of* E *in* S is an element $b \in S$ such that if $y \in E$ then $y \le b$.
- A *lower bound of* E *in* S is an element $l \in S$ such that if $y \in E$ then $l \le y$.
- A *greatest lower bound of* E *in* S is an element $\inf(E) \in S$ such that

  (a) $\inf(E)$ is a lower bound of E in S, and
  (b) If $l \in S$ is a lower bound of E in S then $l \le \inf(E)$.

- A *least upper bound of* E *in* S is an element $\sup(E) \in S$ such that

(a) $\sup(E)$ is a upper bound of E in S, and

(b) If $b \in S$ is a upper bound of E in S then $\sup(E) \le b$.

- The set E is *bounded in S* if E has both an upper bound and a lower bound in S.

Proposition 5.4. *Let S be a poset and let E be a subset of S. If $\sup(E)$ exists then $\sup(E)$ is unique.*

5.3 The subspace lattice $\mathbb{G}(\mathbb{F}^n)$

The *subspace lattice* of $\mathbb{F}^n$ is

$$\mathbb{G}(\mathbb{F}^n) = \{\mathbb{F}\text{-subspaces of } \mathbb{F}^n\} \qquad \text{partially ordered by inclusion.}$$

Then $\mathbb{G}(\mathbb{F}^n)$ is a ranked modular lattice

$$\mathbb{G}(\mathbb{F}^n) = \bigsqcup_{k=0}^{n} \mathbb{G}(\mathbb{F}^n)_k, \qquad \text{where} \quad \mathbb{G}(\mathbb{F}^n)_k = \{\mathbb{F}\text{-subspaces } V \subseteq \mathbb{F}^n \text{ with } \dim(V) = k\}.$$

Theorem 5.5. *Let $\mathbb{F}_q$ be a finite field with q elements. For $r \in \mathbb{Z}_{>0}$ let*

$$[r] = \frac{q^r - 1}{q - 1}, \qquad [r]! = [r][r-1]\cdot[2][1], \qquad \textit{and let} \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!},$$

for $k \in \{0, 1, \ldots, n\}$. Then

$$\mathrm{Card}(\mathbb{G}(\mathbb{F}_q^n)_k) = \begin{bmatrix} n \\ k \end{bmatrix}$$

and

$$\sum_{k=0}^{n} x^k q^{\frac{1}{2}k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix} = (1+x)(1+xq)\cdots(1+xq^{n-1}) = (-x;q)_n.$$

5.3.1 Automorphisms of $\mathbb{G}(\mathbb{F}^n)$

A *morphism of posets* is a function $f\colon P \to Q$ such that P and Q are posets and

$$\text{if } x, y \in P \text{ and } x \le y \quad \text{then} \quad f(x) \le f(y).$$

An *isomorphism* of posets is a morphism $f\colon P \to Q$ such that the inverswe function $f^{-1}\colon Q \to P$ exists and f^{-1} is a morphism of posets.

A *automorphism* of P is an isomorphism $f\colon P \to P$ of posets.

Proposition 5.6. *Let $\mathbb{F}$ be a field.*

$$Aut(\mathbb{G}(\mathbb{F}^n)) = GL_n(\mathbb{F}),$$

where $GL_n(\mathbb{F}) = \{g \in M_n(\mathbb{F}) \mid g^{-1} \text{ exists in } M_n(\mathbb{F})\}$.

HW:. Give an example of a morphism $f\colon P \to Q$ of finite posets that is bijective and is not an isomorphism of posets.

PICTURE

5.3.2 Projective space and cosets

Let $\mathbb{F}$ be a field and define an equivalence relation on $\mathbb{F}^n - \{(0,\ldots,0)\}$ by

$$[a_1,\ldots,a_n] = [\lambda a_1,\ldots,\lambda a_n], \qquad \text{if } a_1,\ldots,a_n \in \mathbb{F} \text{ and } \lambda \in \mathbb{F}^\times.$$

The *projective space* $\mathbb{P}^{n-1}$ is

$$\mathbb{P}^{n-1} = \{\text{equivalence classes}\}.$$

Let $\{e_1,\ldots,e_n\}$ be an $\mathbb{F}$-basis of $\mathbb{F}^n$ and let

$$E = (0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = \mathbb{F}^n), \qquad \text{where} \quad E_k = FF\text{-span}\{e_1,\ldots,e_k\},$$

for $k \in \{0,\ldots,n\}$. Let

$$B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\} \qquad \text{and} \qquad P_k = ????$$

for $k \in \{1,\ldots,n\}$.

Proposition 5.7. *Let $G = GL_n(\mathbb{F}^n) = \mathrm{Aut}(\mathbb{G}(\mathbb{F}^n))$ acting on $\mathbb{G}(\mathbb{F}^n)$.*

(a)

$$\mathrm{Stab}_G(E_k) = P_k \qquad \textit{and} \qquad \mathrm{Stab}_G(E) = B.$$

(b)

$$\mathbb{G}(\mathbb{F}^n)_k \cong G/P_k \qquad \textit{and} \qquad \mathcal{F}(\mathbb{G}(\mathbb{F}^n)) \cong G/B,$$

where $\mathcal{F}(\mathbb{G}(\mathbb{F}^n)) = \{$maximal chains in $\mathbb{G}(\mathbb{F})\}$.

(c)

$$\mathbb{G}(\mathbb{F}^n)_1 \cong \mathbb{P}^{n-1} \quad \textit{and} \quad \mathbb{G}(\mathbb{F}^n)_{n-1} \cong \mathbb{P}^{n-1} \quad \textit{and} \quad \mathbb{G}(\mathbb{F}^n)_k \cong \mathbb{G}(\mathbb{F}^n)_{n-k}.$$

5.3.3 Counting and the Hecke algebra

Let $\mathbb{F}_q$ be a finite field with q elements.

Proposition 5.8.

(a)

$$\mathrm{Card}(GL_n(\mathbb{F}_q^n)) = [n]! q^{\frac{1}{2}n(n-1)}(q-1)^n, \qquad \mathrm{Card}(B) = q^{\frac{1}{2}n(n-1)}(q-1)^n, , \mathrm{Card}(\mathcal{F}(\mathbb{G}(\mathbb{F}_q^n))) = [n]!.$$

(b)

$$\mathrm{Card}(\mathbb{P}^1) = 1+q, \qquad \textit{and} \qquad \mathrm{Card}(\mathbb{P}^{n-1}) = 1+q+\cdots+q^{n-1}.$$

Let $\mathbb{C}[G/B]$ be a vector space with basis indexed by the element of $\mathcal{F}(\mathbb{G}(\mathbb{F}_q^n))$. The group $G = GL_n(\mathbb{F}_q)$ acts on $\mathbb{C}[G/B]$ by the $\mathbb{C}$-linear maps given by

$$g(0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = (0 \subsetneq gV_1 \subsetneq \cdots \subsetneq gV_n), \qquad \text{for } g \in GL_n(\mathbb{F}_q).$$

Theorem 5.9. *For $I \in \{1,\ldots,n-1\}$ define a $\mathbb{C}$-linear map $T_i\colon \mathbb{C}[G/B] \to \mathbb{C}[G/B]$ by*

$$T_i(0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{V_{i-1} \subseteq W \subseteq V_{i+1}} (0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_{i+1} \subsetneq \cdots \subsetneq V_n).$$

Then

$$T_j^2 = (q-1)T_j + q, \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \qquad T_jT_k = T_kT_j, \qquad gT_j = T_jg$$

for $j,k \in \{1,\ldots,n-1\}$ and $i \in \{1,\ldots,n-2\}$ with $k \notin \{j-1,j+1\}$ and $g \in GL_n(\mathbb{F}_q)$.

5.4 The subset lattice $\mathbb{S}(n)$

The *subset lattice* $\mathbb{S}(n)$ is

$$\mathbb{S}(n) = \{\text{subsets of } \{1,\ldots,n\}\}, \qquad \text{partially ordered by inclusion.}$$

The subset lattice $\mathbb{S}(n)$ is a ranked modular lattice with

$$\mathbb{S}(n) = \bigsqcup_{k=0}^{n} \mathbb{S}(n)_k, \qquad \text{wtih} \quad \mathbb{S}(n)_k = \{\text{subsets } V \subseteq \{1,\ldots,n\} \text{ with } \mathrm{Card}(V) = k\}.$$

Then

$$\mathrm{Card}(\mathbb{S}(n)_k) = \binom{n}{k} \qquad \text{and} \qquad \sum_{k=0}^{n} x^k \binom{n}{k} = (1+x)^n$$

is the rank generating function for $\mathbb{S}(n)$.

Proposition 5.10. *The automorphism group of* $\mathbb{S}(n)$ *is the symmetric group*

$$\mathrm{Aut}(\mathbb{S}(n)) = S_n.$$

Proposition 5.11. *For* $k \in \{1,\ldots,n\}$ *let* $E_k = \{1,\ldots,k\}$. *Then* $E_k \in \mathbb{S}(n)$ *and* $(\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_n)$ *is a maximal chain in* $\mathbb{S}(n)$.

$$\mathrm{Stab}_{S_n}(E_k) = S_k \times S_{n-k} \qquad \textit{and} \qquad \mathrm{Stab}_{S_n} = \{1\} = S_1 \times \cdots \times S_1.$$

$$\mathbb{S}(n)_k \cong \frac{S_n}{S_k \times S_{n-k}} \qquad \textit{and} \qquad \mathcal{F}(\mathbb{S}(n)) \cong S_n/\{1\} = S_n.$$

$$\mathbb{S}(n)_k \cong \mathbb{S}(n)_{n-k}.$$

5.4.1 Maximal chains in $\mathbb{S}(n)$

Proposition 5.12. . *The map*

$$\begin{matrix} \mathcal{F}(\mathbb{S}(n)) & \longrightarrow & S_n \\ (\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) & \longmapsto & (V_1, V_2 - V_1, \ldots, V_n - V_{n-1}) \end{matrix} \qquad \textit{is a bijection.}$$

Let $\mathbb{C}S_n$ be the vector space with basis indexed by the elements of $\mathcal{F}(\mathbb{S}(n))$. For $i \in \{1,\ldots,n-1\}$ define a $\mathbb{C}$-linear transformation $s_i \colon \mathbb{C}S_n \to \mathbb{C}S_n$ by

$$s_i(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{V_{i-1} \subsetneq W \subsetneq V_{i+1}} (\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_{i+1} \subsetneq \cdots \subsetneq V_n).$$

Then

$$s_i^2 = 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_i s_j = s_j s_i \ \text{ if } j \notin \{i-1, i+1\},$$

and

$$s_i g = g s_i, \qquad \text{for } g \in S_n,$$

where

$$g(\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = (\emptyset \subsetneq gV_1 \subsetneq \cdots \subsetneq gV_n) \qquad \text{for } g \in S_n \text{ and } (\emptyset \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) \in \mathcal{F}(\mathbb{S}(n)).$$

5.4.2 Simple reflections

Let S_n be the symmetric group of permutation matrices and let

$$s_i = 1 + E_{i,i+1} + E_{i+1,i} - E_{ii} - E_{i+1,i+1}, \qquad \text{for } i \in \{1, \dots, n-1\}.$$

Theorem 5.13. *The symmetric group S_n is presented by generators $s_1, \dots, s_{n-1}$ and relations*

$$s_j^2 = 1, \qquad s_i s_{is+1} s_i = s_{i+1} s_i s_{i+1}, \qquad s_j s_k = s_k s_j,$$

for $j, k \in \{1, \dots, n-1\}$ with $k \notin \{j-1, j+1\}$ and $i \in \{1, \dots, n-2\}$.

Proof. The proof requires four steps:

(1) Generators A in terms of generators B.

(2) Generators B in terms of generators A.

(3) Relations A from relations B.

(4) Relations B from relations A.

Here

Generators A: { permutation matrices}

Relations A: { matrix multiplication of permutation matrices}

Generators B: { simple transpositions}

Relations B: { the braid relations in the statement }

□

5.4.3 Reduced words

Let $w \in S_n$. A reduced word for w is an expression $w = s_{i_1} \cdots s_{i_\ell}$ with $i_1, \dots, i_\ell \in \{1, \dots, n-1\}$ and ℓ minimal. The *length* of w is $\ell(w)$, the length of a reduced word for w.

Let $w \in S_n$. The following is an explicit algorithm for producing a reduced word for w. Let $j_1 > 1$ be minimal such that $w_{j,1} \neq 0$. If j_1 does not exist set $w^{(1)} = w$ and if j_1 does exist set

$$w^{(1)} = s_1 \cdots s_{j_1-1} w.$$

Let $j_2 > 2$ be minimal such that $w^{(1)}_{j,2} \neq 0$. If j_2 does not exist set $w^{(2)} = w^{(1)}$ and if j_2 does exist set

$$w^{(2)} = s_2 \cdots s_{j_2-1} w^{(1)}.$$

Continue this process to produce $w^{(1)}, \dots, w^{(n)}$. Then $w^{(n)} = 1$ and

$$w = \cdots (s_{j-2-1} \cdots s_2)(s_{j_1-1} \cdots s_1) \quad \text{is a reduced word for } w.$$

Proposition 5.14. *Let*

$$\mathrm{Inv}(w) = \{(i, j) \mid i, j \in \{1, \dots, n\} \text{ with } i < j \text{ and } w(i) > w(j)\}.$$

Then $\ell(w) = \mathrm{Card}(\mathrm{Inv}(w))$.

Define a graph $\Gamma(w)$ with

Vertices: $\{$reduced words of $w\}$
Edges: $u \to u'$ if $u' = s_{i_1} \cdots s_{i_\ell}$ is obtained from $u = s_{j_1} \cdots s_{j_\ell}$ by applying a relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ or a relation $s_i s_j = s_j s_i$ with $j \not\in \{i-1, i+1\}$.

Theorem 5.15. *Let $w \in S_n$, The graph $\Gamma(w)$ of reduced words of w is connected.*

Proof. Let

$$w = s_{i_1} \cdots s_{i_\ell} \qquad \text{and} \qquad w = s_{j_1} \cdots s_{j_\ell}$$

be reduced words.

Case 1: $i_1 = j_1$. The two reduced words for w have the same first letter. By induction, the reduced words $v = s_{i_2} \cdots s_{i_\ell}$ and $v = s_{j_2} \cdots s_{j_\ell}$ are connected.

Case 2: $i_1 \neq j_1$. Since $\ell(s_{j_1 w}) < \ell(w)$ then there exists k such that $s_{j_1} w = s_{i_1} \cdots s_{i_{k-1}} \cancel{s_{i_k}} s_{i_{k+1}} \cdots s_{i_\ell}$.

Case 2a: $k \neq \ell$. Then

$$\begin{aligned} &w = s_{j_1} \cdots s_{j_\ell} \\ &w = s_{j_1} s_{i_1} \cdots s_{i_{k-1}} \cancel{s_{i_k}} s_{i_{k+1}} \cdots s_{i_\ell} \quad \text{and} \\ &w = s_{i_1} \cdots s_{i_\ell} \end{aligned}$$

are all reduced words for w. Since the first factor is the same in the first two of these they are connected. Since the last factor is the same in the last two of these they are connected. So, by transitivity, the first is connected to the last.

Case 2b: $k = \ell$ and $j_1 \not\in \{i_1 - 1, i_1 + 1\}$. Then

$$\begin{aligned} &w = s_{j_1} \cdots s_{j_\ell}, \\ &w = s_{j_1} s_{i_1} \cdots s_{i_{\ell-1}}, \\ &w = s_{i_1} s_{j_1} \cdots s_{i_{\ell-1}} \quad \text{and} \\ &w = s_{i_1} s_{i_2} \cdots s_{i_\ell} \end{aligned}$$

and the first two are connected since they have the same first letter, the middle two are connected by the move $s_{j_1} s_{i_1} = s_{j_1} s_{i_1}$ and the last two are connected since they have the same first letter.

Case 2c: $k = \ell$ and $j_1 \in \{i_1 - 1, i_1 +\}$. Then

$$\begin{aligned} &w = s_{i_1} s_{i_2} \cdots s_{i_\ell}, \\ &w = s_{i_1} s_{j_1} s_{i_1} \cdots s_{i_{r-1}} \cancel{s_{i_r}} s_{i_{r+1}} \cdots s_{i_{\ell-1}}, \\ &w = s_{j_1} s_{i_1} s_{j_1} \cdots s_{i_{r-1}} \cancel{s_{i_r}} s_{i_{r+1}} \cdots s_{i_{\ell-1}}, \quad \text{and} \\ &w = s_{j_1} s_{j_2} \cdots s_{j_\ell}, \end{aligned}$$

and the first two are connected since they have the same first letter, the middle two are connected by the move $s_{i_1} s_{j_1} s_{i_1} = s_{j_1} s_{i_1} s_{j_1}$ and the last two are connected since they have the same first letter. $\square$

6 Week 3: Generating symmetric functions

6.1 Generating function definitions

Define

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) \qquad \text{and} \qquad (a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots.$$

Define $g_r = g_r(x;q,t), q_r = q_r(x;t), h_r = h_r(x), e_r = e_r(x)$ by the generating functions

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} g_r z^r, \qquad \prod_{i=1}^n \frac{1-tx_iz}{1-x_iz} = \sum_{r\in\mathbb{Z}_{\geq 0}} q_r z^r,$$

$$\prod_{i=1}^n \frac{1}{1-x_iz} = \sum_{r\in\mathbb{Z}_{\geq 0}} h_r z^r, \qquad \prod_{i=1}^n (1+x_iz) = \sum_{r\in\mathbb{Z}_{\geq 0}} e_r z^r,$$

Remark 6.1. In later sections we will understand that the g_r are, up to a normalization factor, the Macdonald polynomials for a single row, the q_r are Hall-Littlewood polynomials for a single row, and the h_r are Schur functions for a single row. In formulas

$$\begin{aligned}
g_r &= \frac{(t;q)_r}{(q;q)_r} P_{(r)}(x;q,t), && \text{one row Macdonald polynomials,} \\
q_r &= (1-t)P_{(r)}(x;0,t), && \text{one row Hall-Littlewood polynomials,} \\
h_r &= s_{(r)}(x), && \text{one row Schur functions, and} \\
e_r &= P_{(1^r)}(x;q,t) && \text{one column Macdonald polynomials,} \\
&= P_{(1^r)}(x;0,t) && \text{one column Hall-Littlewood polynomials,} \\
&= s_{(1^r)}(x) && \text{one column Schur functions.} \quad \square
\end{aligned}$$

Let us extend these definitions just slightly by defining $\tilde{g}_r = \tilde{g}_r(x;q,t,u)$ and $\tilde{q}_r = \tilde{q}_r(x;t,u)$ by

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{g}_r z^r \qquad \text{and} \qquad \prod_{i=1}^n \frac{1-tx_iz}{1-ux_iz} = \sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{q}_r z^r.$$

This is not really an extension since $g_r(x;q,t) = \tilde{g}_r(x;q,t,1)$ and

$$\tilde{g}_r(x_1,\ldots,x_n;q,t,u) = u^r\tilde{g}_r(u^{-1}x_1,\ldots,u^{-1}x_n;q,t,u) = u^r g_r(x;q,tu^{-1}),$$

so that any formula for $\tilde{g}_r$ immediately converts to a formula for g_r and vice versa. From the generating function definitions,

$$\begin{aligned}
\tilde{q}_r(x;t,u) &= \tilde{g}_r(x;0,t,u), & q_r(x;t) &= \tilde{g}_r(x;0,t,1), \\
h_r(x) &= \tilde{g}_r(x;0,0,1), & e_r(x) &= \tilde{g}_r(x;0,-1,0).
\end{aligned} \tag{6.1}$$

6.2 Formulas in terms of power sums

The *power sums* $p_r \in \mathbb{C}[x_1,\ldots,x_n]$, for $r\in\mathbb{Z}_{\geq 0}$, are defined by

$$p_0 = 1 \qquad \text{and} \qquad p_r = x_1^r + \cdots + x_n^r \quad \text{for } r\in\mathbb{Z}_{>0}.$$

For a sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}.$$

Since

$$\log(1-z) = \int \frac{-1}{1-z}\, dz = \int -(1+z+z^2+\cdots)\, dz = -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \cdots = -\sum_{r\in\mathbb{Z}_{>0}} \frac{1}{r} z^r$$

then

$$\begin{aligned}
\log\Big(\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty}\Big) &= \sum_{i=1}^n \sum_{\ell\in\mathbb{Z}_{\ge 0}} \big(\log(1-tx_izq^\ell) - \log(1-ux_izq^\ell)\big) \\
&= \sum_{i=1}^n \sum_{\ell\in\mathbb{Z}_{\ge 0}} \sum_{r\in\mathbb{Z}_{>0}} \big(-\frac{1}{r}t^r x_i^r q^{\ell r} z^r + \frac{1}{r} u^r x_i^r q^{\ell r} z^r\big) \\
&= \sum_{\ell\in\mathbb{Z}_{\ge 0}} \sum_{r\in\mathbb{Z}_{>0}} \frac{1}{r}(u^r - t^r) q^{\ell r} p_r z^r = \sum_{r\in\mathbb{Z}_{>0}} \Big(\frac{u^r - t^r}{1-q^r}\Big)\frac{p_r}{r} z^r. \qquad (6.2)
\end{aligned}$$

Define

$$z_\lambda(q,t,u) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{u^{\lambda_i} - t^{\lambda_i}}, \qquad \text{where} \quad z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$$

for $\lambda = (\lambda_1, \ldots, \lambda_n) = (1^{m_1} 2^{m_2} \cdots)$. Taking the exponential of both sides of (6.2) gives

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty} = \sum_{r\in\mathbb{Z}_{\ge 0}} \Big(\sum_{|\lambda|=r} \frac{1}{z_\lambda(q,t,u)} p_\lambda(x)\Big) z^r$$

so that

$$\tilde{g}_r = \sum_{|\lambda|=r} \frac{1}{z_\lambda(q,t,u)} p_\lambda(x) = \sum_{|\lambda|=r} \Big(\prod_{i=1}^{\ell(\lambda)} \frac{u^{\lambda_i} - t^{\lambda_i}}{1-q^{\lambda_i}}\Big) \frac{p_\lambda}{z_\lambda}. \qquad (6.3)$$

Applying (6.1) gives

$$\tilde{q}_r = \sum_{|\lambda|=r} \Big(\prod_{i=1}^{\ell(\lambda)} (u^{\lambda_i} - t^{\lambda_i})\Big)\frac{p_\lambda}{z_\lambda}, \qquad q_r = \sum_{|\lambda|=r} \Big(\prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i})\Big)\frac{p_\lambda}{z_\lambda},$$

$$h_r = \sum_{|\lambda|=r} \frac{1}{z_\lambda} p_\lambda(x), \qquad e_r = \sum_{|\lambda|=r} (-1)^{r-\ell(\lambda)} \frac{p_\lambda}{z_\lambda}.$$

6.3 Generalized Newton identities

Taking the coefficient of z^r on each side of the identity

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(ux_iz;q)_\infty} \prod_{i=1}^n \frac{(ux_iz;q)_\infty}{(sx_iz;q)_\infty} = \prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(sx_iz;q)_\infty}$$

gives

$$\tilde{g}_r(x;q,t,u) + \Big(\sum_{j=1}^{r-1} \tilde{g}_j(x;q,t,u)\tilde{g}_{r-j}(x;q,u,s)\Big) + \tilde{g}_r(x;q,u,s) = \tilde{g}_r(x;q,t,s). \qquad (6.4)$$

Using the specializations in (6.1),

$$\tilde q_r(x;t,u) + \Big(\sum_{j=1}^{r-1} \tilde q_j(x;t,u)\tilde q_{r-j}(x;u,s)\Big) + \tilde q_r(x;u,s) = \tilde q_r(x;t,s),$$

$$\tilde q_r(x;t,u) + \left(\sum_{j=1}^{r-1} h_j(x)u^j\tilde q_{r-j}(x;t,u)\right) - h_r(x)(u^r-t^r) = 0,$$

$$\tilde q_r(x;t,u) + \left(\sum_{j=1}^{r-1} e_j(x)(-t)^j\tilde q_{r-j}(x;t,u)\right) + (-1)^r e_r(x)(u^r-t^r) = 0,$$

$$\sum_{j=0}^{r}(-t)^{r-j}(u^j-t^j)h_j(x)e_{r-j}(x) = (u-t)\tilde q_r(x;t,u),$$

$$r\tilde q_r(x;t,u) - \left(\sum_{j=1}^{r-1} p_j(x)(u^j-t^j)\tilde q_{r-j}(x;t,u)\right) - p_r(x)(u^r-t^r) = 0,$$

Further specializations give the Wronski identities

$$\sum_{i+j=k}(-1)^i e_i h_j = 0 \qquad \text{and} \qquad \sum_{i+j=k}(-1)^i(t^iq^j-1)e_i g_j = 0$$

and the *Newton identities*

$$kh_k = \sum_{i=1} p_i h_{k-i} \qquad \text{and} \qquad ke_k = \sum_{i=1}^{k}(-1)^{i-1}p_i e_{k-i}. \tag{6.5}$$

6.4 Formulas in terms of sequences $(i_1,\ldots,i_r)$

Using the geometric series expansions

$$\frac{1}{1-ux_iz} = 1 + ux_iz + u^2x_i^2z^2 + \cdots$$

gives

$$\frac{1-tx_iz}{1-ux_iz} = 1 + \frac{(u-t)x_iz}{1-ux_iz} = 1 + (u-t)x_iz(1+ux_iz+u^2x_i^2z^2+\cdots),$$

Apply this, factor by factor, to the product

$$\prod_{i=1}^{n}\frac{1-tx_iz}{1-ux_iz} = \Big(\frac{1-tx_1z}{1-ux_1z}\Big)\cdots\Big(\frac{1-tx_nz}{1-ux_nz}\Big)$$

to get

$$\tilde q_r = \sum_{1\le i_1\le\cdots\le i_r\le n}(u-t)^{1+\mathrm{Card}\{j\ \mid\ i_j<i_{j+1}\}}u^{\mathrm{Card}\{j\ \mid\ i_j=i_{j+1}\}}x_{i_1}x_{i_2}\cdots x_{i_r}. \tag{6.6}$$

Dividing $\tilde q_r$ by $(u-t)$ and specializing $t=u$ gives

$$\Big(\frac{1}{u-t}\tilde q_r\Big)\Big|_{t=u} = p_r = \sum_{i_1=i_2=\cdots=i_r} x_{i_1}\cdots x_{i_r}. \tag{6.7}$$

Applying $(1-ux_iz)^{-1} = 1 + ux_iz + u^2x_i^2z^2 + \cdots$ and expanding, factor by factor, the product

$$\prod_{i=1}^n \frac{1-tx_iz}{1-ux_iz} = \Big(\frac{1}{1-ux_1z}\Big)\cdots\Big(\frac{1}{1-ux_nz}\Big)(1-tx_nz)\cdots(1-tx_1z)$$

gives

$$\tilde{q}_r = \sum_{i_1\le i_2\le\cdots\le i_k > i_{k+1}>\cdots>i_r} u^{k-1}(-t)^{r-k}x_{i_1}\cdots x_{i_k}x_{i_{k+1}}\cdots x_{i_r}. \qquad (6.8)$$

Applying (6.1) gives

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}\cdots x_{i_r}, \qquad \text{and} \qquad h_r = \sum_{i_1\le i_2\le\cdots\le i_r} x_{i_1}\cdots x_{i_r}. \qquad (6.9)$$

6.5 Formulas in terms of monomial symmetric functions

For $\lambda = (\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n$ with $\lambda_1\ge\cdots\ge\lambda_n\ge 0$, the *monomial symmetric function* is defined by

$$m_\lambda = \sum_{\gamma\in S_n\lambda} x^\gamma, \qquad \text{where} \quad x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}.$$

Applying the expansion (from the infinite q-binomial theorem, see below)

$$\frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} = \sum_{r\in\mathbb{Z}_{\ge 0}} \frac{(t;q)_r}{(q;q)_r}x_i^rz^r,$$

and expanding the product

$$\prod_{i=1}^n \frac{(tx_iz;q)_\infty}{(x_iz;q)_\infty} = \frac{(tx_1z;q)_\infty}{(x_1z;q)_\infty}\cdots\frac{(tx_nz;q)_\infty}{(x_nz;q)_\infty},$$

gives

$$\tilde{g}_r = \sum_{|\mu|=r} u^r\frac{(tu^{-1};q)_\mu}{(q;q)_\mu}m_\mu, \qquad \text{where} \quad \frac{(tu^{-1};q)_\mu}{(q;q)_\mu} = \frac{(tu^{-1};q)_{\mu_1}\cdots(tu^{-1};q)_{\mu_\ell}}{(q;q)_{\mu_1}\cdots(q;q)_{\mu_\ell}}.$$

if $\mu = (\mu_1,\ldots,\mu_\ell)$. Using the specializations in (6.1),

$$\tilde{q}_r = \sum_{|\mu|=r} u^{r-\ell(\mu)}(u-t)^{\ell(\mu)}m_\mu, \qquad g_r = \sum_{|\mu|=r}\frac{(t;q)_\mu}{(q;q)_\mu}m_\mu, \qquad q_r = \sum_{|\mu|=r}(1-t)^{\ell(\mu)}m_\mu.$$

$$h_r = \sum_{|\mu|=r} m_\mu, \qquad e_r = m_{(1^r)}, \qquad p_r = m_{(r)}.$$

6.6 The Cauchy-Macdonald kernel

For a sequence of nonnegative integers $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ define

$$\tilde{g}_\lambda = \tilde{g}_{\lambda_1}\,\tilde{g}_{\lambda_2}\cdots\tilde{g}_{\lambda_\ell}, \qquad \tilde{q}_\lambda = \tilde{q}_{\lambda_1}\,\tilde{q}_{\lambda_2}\cdots\tilde{q}_{\lambda_\ell}, \qquad h_\lambda = h_{\lambda_1}\,h_{\lambda_2}\cdots h_{\lambda_\ell}, \qquad e_\lambda = e_{\lambda_1}\,e_{\lambda_2}\cdots e_{\lambda_\ell}.$$

Then

$$\begin{aligned}\prod_{i,j}\frac{(tx_iy_j;q)_\infty}{(ux_iy_j;q)_\infty} &= \sum_\lambda \tilde{g}_\lambda(x;q,t,u)m_\lambda(y)\\ &= \sum_\lambda \frac{1}{z_\lambda(q,t,u)}p_\lambda(x)p_\lambda(y).\end{aligned}$$

6.7 Binomial theorems

Using

$$\prod_{i=1}^{n}(u+x_iz) = u^n\prod_{i=1}^{n}(1+x_i\tfrac{z}{u}) = \sum_{r=0}^{n} u^{n-r}z^re_r(x),$$

and

$$\prod_{i=1}^{n}\frac{1}{(u-x_iz)} = u^{-n}\prod_{i=1}^{n}\frac{1}{(1-x_i\frac{z}{u})} = \sum_{r\in\mathbb{Z}_{\ge0}} u^{-n-r}z^rh_r(x),$$

and specializing $x_1 = x_2 = \cdots = x_n = 1$ gives the *binomial theorem*,

$$(u+z)^n = \sum_{r=0}^{n} u^{n-r}z^r\binom{n}{r} \qquad \text{and} \qquad (u-z)^{-n} = \sum_{r\in\mathbb{Z}_{\ge0}} u^{-n-r}z^r\binom{n+r-1}{r},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = e_r(1,1,\ldots,1) \qquad \text{and} \qquad \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!} = h_r(1,1,\ldots,1).$$

Letting $x_i = q^{i-1}$ gives the q-*binomial theorem*,

$$\prod_{i=1}^{n}(u+q^{i-1}z) = \sum_{r=0}^{n} q^{\frac12 r(r-1)}\begin{bmatrix} n\\ r\end{bmatrix}u^{n-r}z^r \qquad \text{and} \qquad \prod_{i=1}^{n}\frac{1}{(u-q^{i-1}z)} = \sum_{r\in\mathbb{Z}_{\ge0}}\begin{bmatrix} n+r-1\\ r\end{bmatrix}u^{-n-r}z^r,$$

where

$$e_r(1,q,q^2\ldots,q^{n-1}) = \frac{(q;q)_n}{(q;q)_r(q;q)_{n-r}} = \begin{bmatrix} n\\ r\end{bmatrix} \qquad \text{and}$$

$$h_r(1,q,q^2\ldots,q^{n-1}) = \frac{(q;q)_{n+r-1}}{(q;q)_r(q;q)_{n-1}} = \begin{bmatrix} n+r-1\\ r\end{bmatrix}.$$

A general infinite q-binomial theorem is

$$\frac{(tz;q)_\infty}{(uz;q)_\infty} = \prod_{i=1}^{\infty}\Big(\frac{1-tq^{i-1}z}{1-uq^{i-1}z}\Big) = \sum_{r\in\mathbb{Z}_{\ge0}}\Big(\prod_{i=1}^{r}\frac{u-tq^{i-1}}{1-q^i}\Big)z^r = \sum_{r\in\mathbb{Z}_{\ge0}} u^r\frac{(tu^{-1};q)_r}{(q;q)_r}z^r, \tag{6.10}$$

A one sentence proof of the infinite q-binomial theorem: Recognize that

$$L(z;q,t,u) = \frac{(tz;q)_\infty}{(uz;q)_\infty} \qquad \text{satisfies the recursion} \qquad L(z;q,t,u) = \frac{(1-tz)}{(1-uz)}L(qz;q,t,u)$$

which provides a recursion on the coefficients of $L(z;q,tu) = \sum_{r\in\mathbb{Z}_{\ge0}} c_r(q,t,u)z^r$ as

$$c_r(q,t,u)q^r - tc_{r-1}(q,t,u)q^{r-1} = c_r(q,t,u) - uc_{r-1}(q,t,u).$$

so that

$$c_r(q,t,u) = c_{r-1}(q,t,u)\frac{u-tq^{r-1}}{1-qq^{r-1}} = u^r\frac{(tu^{-1};q)_r}{(q;q)_r}.$$

Specializing $t = 0$ and $u = 0$ in (6.10) give

$$\prod_{i=1}^{\infty}(1+q^{i-1}z) = (-z,q)_\infty = \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{1\cdot q\cdot q^2\cdots q^{r-1}}{(q,q)_r} z^r \qquad \text{and}$$

$$\prod_{i=1}^{\infty}\frac{1}{(1-q^{i-1}z)} = \frac{1}{(z,q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{1}{(q,q)_r} z^r.$$

The finite q-binomial theorem is obtained from (6.10) by putting $t = q^n$ and $u = 1$ so that the left hand side becomes

$$\frac{(q^n z;q)_\infty}{(z;q)_\infty} = \frac{1}{(z,q)_n} = \prod_{i=1}^{n}\frac{1}{1-q^{i-1}z}$$

and the right hand side is

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \frac{(q^n;q)_r}{(q;q)_r} z^r, \qquad \text{with} \quad (q^n;q)_r \frac{1}{(q;q)_r} = \frac{(q;q)_{n+r-1}}{(q;q)_{n-1}}\cdot\frac{1}{(q;q)_r} = \begin{bmatrix} n+r-1 \\ r \end{bmatrix},$$

so that

$$\prod_{i=1}^{n}\frac{1}{1-q^{i-1}z} = \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{(q;q)_{n+r-1}}{(q;q)_r(q;q)_{n-1}} z^r = \sum_{r\in\mathbb{Z}_{\geq 0}} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} z^r.$$

6.8 Monomial expansion of $\tilde{g}_\lambda, \tilde{q}_\lambda, h_\lambda$ and e_λ

For a sequence of nonnegative integers $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ define

$$\tilde{g}_\lambda = \tilde{g}_{\lambda_1}\tilde{g}_{\lambda_2}\cdots\tilde{g}_{\lambda_\ell}, \qquad \tilde{q}_\lambda = \tilde{q}_{\lambda_1}\tilde{q}_{\lambda_2}\cdots\tilde{q}_{\lambda_\ell}, \qquad h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_\ell}, \qquad e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_\ell}.$$

For an $n\times\ell$ matrix $a = (a_{ij})$ with entries from $\mathbb{Z}_{\geq 0}$ let

$$\begin{matrix} rs(a) = (\mu_1,\ldots,\mu_n), \\ cs(a) = (\lambda_1,\ldots,\lambda_\ell), \end{matrix} \qquad \text{where} \qquad \mu_i = \sum_{j=1}^{\ell} a_{ij} \quad \text{and} \quad \lambda_j = \sum_{i=1}^{n} a_{ij},$$

so that $rs(a)$ and $cs(a)$ are the sequences of row sums and column sums of a, respectively. Define

$$x^a = x^{rs(a)} = \prod_{i=1}^{n}\prod_{j=1}^{\ell}(x_i)^{a_{ij}}, \qquad y^a = y^{cs(a)} = \prod_{j=1}^{\ell}\prod_{i=1}^{n}(y_j)^{a_{ij}}, \qquad \mathrm{wt}_{q,u,t}(a) = \prod_{j=1}^{\ell}\prod_{i=1}^{n} u^{a_{ij}}\frac{(tu^{-1};q)_{a_{ij}}}{(q;q)_{a_{ij}}},$$

where, by definition, $(a;q)_0 = 1$. For a sequence $\mu = (\mu_1,\ldots,\mu_n)$ of nonnegative integers let

$$A_{\mu\lambda} = \{a\in M_{n\times\ell}(\mathbb{Z}_{\geq 0}) \mid cs(a) = \lambda,\ rs(a) = \mu\}.$$

Then

$$\tilde{g}_\lambda = \sum_{\mu} a_{\mu\lambda}(q,t)m_\mu\ , \qquad \text{where} \qquad a_{\mu\lambda}(q,t) = \sum_{a\in A_{\mu\lambda}} \mathrm{wt}_{q,t,u}(a),$$

and the first sum is over partitions μ such that $|\mu| = |\lambda|$.

7 Week 4: Crystals and RSK

7.1 The category of crystals

For $i \in \{1, \ldots, n\}$ let $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith spot. Let

$$B(\square) = \big(1 \xrightarrow{\tilde{f}_1} 2 \xrightarrow{\tilde{f}_1} \cdots \xrightarrow{\tilde{f}_1} n\big) \qquad \text{with} \quad \mathrm{wt}(i) = \varepsilon_i.$$

A *crystal* is an element B of the category generated by $B(\square)$ under direct sums and tensor products. A crystal is a (finite) set B with functions

$$\mathrm{wt}\colon B \to \mathbb{Z}^n \qquad \text{and} \qquad \tilde{f}_i\colon B \to B \cup \{0\}, \quad \text{for } i \in \{1, \ldots, n-1\}.$$

The *crystal graph* of B is the labeled graph with

$$\text{Vertices: } B \qquad \text{and} \qquad \text{Labeled edges: } \ b \xrightarrow{\tilde{f}_i} \tilde{f}_i b.$$

A *crystal morphism* from B_1 to B_2 is a function $\Phi\colon B_1 \to B_2$ such that

$$\mathrm{wt}(\Phi(b)) = \mathrm{wt}(b) \qquad \text{and} \qquad \tilde{f}_i(\Phi(b)) = \Phi(\tilde{f}_i b),$$

for $b \in B_1$ and $i \in \{1, \ldots, n-1\}$. The *character* of a crystal B is

$$\mathrm{char}(B) = \sum_{p \in B} x^{\mathrm{wt}(p)}, \qquad \text{where} \quad x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$$

if $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$. The *direct sum* of crystals B_1 and B_2 is

$$B_1 \oplus B_2 = B_1 \sqcup B_2 \qquad \text{with wt and } \tilde{f}_i \text{ inherited from } B_1 \text{ and } B_2.$$

For $i \in \{1, \ldots, n-1\}$ define

$$\tilde{e}_i\colon B \to B \cup \{0\} \qquad \text{by} \quad \tilde{e}_i(\tilde{f}_i b) = b \ \text{ if } \tilde{f}_i b \neq 0,$$

and $\tilde{e}_i b = 0$ if there does not exist $b' \in B$ such that $b = \tilde{f}_i b'$. Let $b \in B$ and $i \in \{1, \ldots, n-1\}$. Define $d_i^+(b)$ and $d_i^-(b)$ by

$$\begin{array}{ccc} \tilde{e}_i^{d_i^+(b)} b \neq 0 & \text{and} & \tilde{e}_i^{d_i^+(b)+1} b = 0, \\ \tilde{f}_i^{d_i^-(b)} b \neq 0 & \text{and} & \tilde{f}_i^{d_i^-(b)+1} b = 0. \end{array}$$

Then

$$\tilde{e}_i^{d_i^+(b)} b \xrightarrow{\tilde{f}_i} \cdots \xrightarrow{\tilde{f}_i} \tilde{e}_i b \xrightarrow{\tilde{f}_i} b \xrightarrow{\tilde{f}_i} \tilde{f}_i b \xrightarrow{\tilde{f}_i} \cdots \xrightarrow{\tilde{f}_i} \tilde{f}_i^{d_i^-(b)} b$$

is the *i-string of b*.

The *tensor product* of crystals B_1 and B_2 is

$$B_1 \otimes B_2 = B_1 \times B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$$

with

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$$

and

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } d_i^+(b_1) > d_i^-(b_2), \\ b_1 \otimes \tilde{f}_i b_2, & \text{if } d_i^+(b_1) \le d_i^-(b_2), \end{cases}$$

Then

$$\mathrm{char}(B_1 \oplus B_2) = \mathrm{char}(B_1) + \mathrm{char}(B_2) \qquad \text{and} \qquad \mathrm{char}(B_1 \otimes B_2) = \mathrm{char}(B_1)\mathrm{char}(B_2)$$

and

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } d_i^+(b_1) \ge d_i^-(b_2), \\ b_1 \otimes \tilde{e}_i b_2, & \text{if } d_i^+(b_1) < d_i^-(b_2), \end{cases}$$

HW: Show that if B_1, B_2, B_3 are crystals then

$$\begin{matrix} \Phi\colon & (B_1 \otimes B_2) \otimes B_3 & \longrightarrow & B_1 \otimes (B_2 \otimes B_3) \\ & b_1 \otimes b_2 \otimes b_3 & \longmapsto & b_1 \otimes b_2 \otimes b_3 \end{matrix}$$

is a crystal isomorphism.

- A *subcrystal* of B is a subset of B closed under the operators $\tilde{e}_i$ and $\tilde{f}_i$ (for $i \in \{1, \ldots, n-1\}$).
- A crystal is *irreducible*, or *simple*, if B has no subcrystals except $\emptyset$ and B.

A *highest weight element* of a crystal B is $b \in B$ such that

$$\text{if } i \in \{1, \ldots, n-1\} \quad \text{then} \quad \tilde{e}_i b = 0.$$

Let

$$B^+ = \{\text{highest weight elements of } B\} \qquad \text{and let} \qquad B^+_\lambda = \{b \in B^+ \mid \mathrm{wt}(b) = \lambda\},$$

for $\lambda \in \mathbb{Z}^n$.

Theorem 7.1.

(a) A crystal B is irreducible if and only if the crystal graph of B is connected.

(b) A crystal B is irreducible if and only if $\mathrm{Card}(B^+) = 1$.

Proposition 7.2. *Assume B_1 and B_2 are irreducible crystals.*

(a) If $\Phi\colon B_1 \to B_2$ is a crystal morphism then Φ is a crystal isomorphism, $B_1 \cong B_2$.

(b) If $B_1^+ = \{b_1^+\}$ and $B_2^+ = \{b_2^+\}$ then

$$B_1 \cong B_2 \quad \textit{if and only if} \quad \mathrm{wt}(b_1^+) = \mathrm{wt}(b_2^+).$$

Theorem 7.3. *Two crystals B_1 and B_2 are isomorphic if and only if*

$$\mathrm{char}(B_1) = \mathrm{char}(B_2).$$

Proof. Decompose B_1 and B_2 into connected components. Let $B(\lambda)$ be the irreducible crystal of highest weight λ,

$$B(\lambda)^+ = \{b_\lambda^+\} \qquad \text{and} \qquad \mathrm{wt}(b_\lambda^+) = \lambda.$$

Then

$$B_1 \cong \bigsqcup_{p \in B_1^+} B(\mathrm{wt}(p)) \cong B_2.$$

So

$$\mathrm{char}(B_1) = \sum_{p \in B_1^+} \mathrm{char}(B(\mathrm{wt}(p))) = \mathrm{char}(B_2).$$

□

7.2 The crystal of words $B^{\otimes k}$

Knuth Equivalence.

(a) If $x \le y < z$ then $xz \cdot y \to z \cdot xy$,

(b) If $x < y \le z$ then $yz \cdot x \to y \cdot xz$.

We need a story for this: z doesn't want to be near y?? Check also Fulton's Young tableaux book.

7.3 The Weyl character formula

For $i \in \{1, \ldots, n-1\}$ define $s_i \colon \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ by

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \ldots, x_n).$$

A *symmetric function* is an element of

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid \text{if } w \in S_n \text{ then } wf = f\}.$$

Let B be a crystal. Let $b \in B$ and $i \in \{1, \ldots, n-1\}$. The *i-stirng* of b is

$$\{\tilde{f}_i^{d_i^-(b)} b, \ldots, \tilde{f}_i b, b, \tilde{e}_i b, \ldots, \tilde{e}_i^{d_i^+(b)} b\} = S_i(b).$$

Let $s_i p$ be the element of $S_i(b)$ such that

$$\mathrm{wt}(s_i b) = s_i \mathrm{wt}(b).$$

Then $s_i(s_i(p)) = p$ and

$$s_i \mathrm{char}(B) = \sum_{b \in B} x^{s_i \mathrm{wt}(b)} = \sum_{b \in B} x^{\mathrm{wt}(s_i b)} = \mathrm{char}(B).$$

So char(B) is a symmetric function.
MORE HERE MORE HERE

Corollary 7.4. *Let λ be a partition. Then*

$$\mathrm{char}(B(\lambda)) = s_\lambda.$$

7.4 The crystals $B(\lambda)$

7.5 HW for Crystals and RSK

1. Use RSK to prove the three Cauchy identities for Schur functions.

2. Use RSK to prove the Pieri rules:

$$e_r s_\lambda, \quad h_r s_\lambda, \quad q_r s_\lambda, \quad p_r s_\lambda.$$

3. Use RSK to prove the Murnaghan-Nakayama rules for symmetric group and Hecke algebra characters,

$$p_\mu = \sum_{\lambda \in Y_k} \chi_{S_k}^\lambda(\mu) s_\lambda \qquad \text{and} \qquad q_\mu = \sum_{\lambda \in Y_k} \chi_{H_k}^\lambda(\mu) s_\lambda.$$

4. Use RSK to prove

$$n! = \sum_{\lambda} f_\lambda^2, \qquad (x_1 + \cdots + x_n)^k = \sum_{\lambda\in Y_k} f_\lambda s_\lambda, \qquad n^k = \sum_{\lambda\in Y_k} f_\lambda d_\lambda.$$

5. Use RSK to prove that

$$\sum_{\lambda\in Y_k} f_\lambda = \#\{\text{involutions in } S_k\}.$$

6. Prove that if $w \in S_k$ and $RSK(w) = (P,Q)$ then $RSK(w^{-1}) = (Q,P)$.

8 Week 5: Products of symmetric functions

8.1 Tensor products and restrictions

Theorem 8.1.

$$s_\mu s_\nu = \sum_{\substack{q\in B(\nu)\\ p_\mu^+\otimes q\subseteq C-\rho}} s_{\mu+\mathrm{wt}(q)} \qquad \textit{and} \qquad s_\lambda = \sum_{\substack{p\in B(\lambda)\\ p\in C_J-\rho_J}} s^J_{\mathrm{wt}(p)}.$$

8.2 The combinatorial R-matrix and RSK

Let

$$f_\lambda = \mathrm{Card}(\hat{S}_n^\lambda) = \#\{\text{standard tableaux of shape } \lambda\},$$
$$d_\lambda = \mathrm{Card}(B(\lambda)) = \#\{\text{SSYTs of shape } \lambda \text{ filled from } \{1,\ldots,n\}\},$$
$$s_\lambda(x) = s_\lambda(x_1,\ldots,x_t), \qquad \text{and} \qquad s_\lambda(y) = s_\lambda(y_1,\ldots,y_s)$$

Theorem 8.2.

$$n! = \sum_{\lambda\in\mathbb{Y}_n} f_\lambda^2, \qquad n^k = \sum_{\lambda\in\mathbb{Y}_n} f_\lambda d_\lambda, \qquad (x_1+\cdots+x_n)^k = \sum_{\lambda\in\mathbb{Y}_n} f_\lambda s_\lambda.$$

$$\prod_{j=1}^s\prod_{i=1}^t \frac{1}{1-x_iy_j} = \sum_{\ell(\lambda)\le\min(s,t)} s_\lambda(x)s_\lambda(y), \qquad \prod_{j=1}^s\prod_{i=1}^t \frac{1}{1+x_iy_j} = \sum_{\lambda\subseteq(s^t)} s_\lambda(x)s_{\lambda'}(y).$$

For each of these there are three nice proofs:

(a) by RSK insertion,

(b) by crystals,

(c) by double centralizer algebras.

The corresponding cateogrifications are

$$\begin{array}{ccccc}
n! & \longleftrightarrow & S_n & \longleftrightarrow & \mathbb{C}S_n \\
(x_1+\cdots+x_n)^k & \longleftrightarrow & B(\square)^{\otimes k} & \longleftrightarrow & L(\square)^{\otimes k} \\
\prod \frac{1}{1-x_iy_j} & \longleftrightarrow & M_{t\times s}(\mathbb{Z}_{\ge 0}) & \longleftrightarrow & S(V_s\otimes V_t) \\
\prod(1+x_iy_j) & \longleftrightarrow & M_{t\times s}(\{0,1\}) & \longleftrightarrow & \Lambda(V_s\otimes V_t)
\end{array}$$

8.3 Pieri rules

Let $n \in \mathbb{Z}_{>0}$ and let λ, μ be partitions of n. Define

$$\chi^\lambda_{q,t}(\mu) = \sum_{Q\in \hat{S}^\lambda_n} \mathrm{rwt}^\mu_{q,t}(Q),$$

where

$$\hat{S}^\lambda_n = \{\text{standard tableaux of shape } \lambda\}$$

$$\mathrm{rwt}^\mu_{q,t}(Q) = \prod_{\substack{j\in Q \\ j\notin J(\mu)}} f_\mu(j;Q), \qquad \text{with} \quad J(\mu) = \{\mu_1, \mu+1\mu_2, \ldots, \mu_1+\cdots\mu_\ell\}$$

and

$$f_\mu(j;q) = \begin{cases} -t, & \text{if } j+1 \text{ is sw of } j \text{ in } Q, \\ 0, & \text{if } j+1\notin J(\mu) \text{ and} \\ & \quad j+1 \text{ is ne of } j \text{ in } Q \text{ and} \\ & \quad j+2 \text{ is sw of } j+1 \text{ in } Q, \\ q, & \text{otherwise.} \end{cases}$$

Define

$$\chi^\lambda_{H_n}(\mu) = \chi^\lambda_{q,1}(\mu) \qquad \text{and} \qquad \chi^\lambda_{S_n}(\mu) = \chi^\lambda_{1,1}(\mu).$$

These are the characters of the Hecke algebra and the characters of the symmetric group, respectively.

Define

$$\tilde{q}_r = \tilde{q}_k(x_1,\ldots,x_n;q,t) = \sum_{i_1\le\cdots\le i_r>i_{r+1}>\cdots>i_k} q^{r-1}(-t)^{k-r}x_{i_1}\cdots x_{i_k}$$

and let

$$h_r = \tilde{q}_r(x_1,\ldots,x_n;1,0), \quad e_r = \tilde{q}_r(x_1,\ldots,x_n;0,-1), \quad p_r = \Big(\frac{1}{t-q}\tilde{q}_r(x_1,\ldots,x_n;q,t)\Big)\Big]_{t=q}.$$

Theorem 8.3. *Let μ be a partition. Then*

$$h_r s_\mu = \sum_{\lambda/\mu \text{ hs length } r} s_\lambda \qquad (\textit{sum over horizontal strips } \lambda/\mu \textit{ of length } r)$$

$$e_r s_\mu = \sum_{\lambda/\mu \text{ vs length } r} s_\lambda \qquad (\textit{sum over vertical strips } \lambda/\mu \textit{ of length } r)$$

$$p_r s_\mu = \sum_{\lambda/\mu \text{ bs length } r} (-1)^{\mathrm{ht}(\lambda/\mu)} s_\lambda \qquad (\textit{sum over border strips } \lambda/\mu \textit{ of length } r)$$

$$\tilde{q}_r s_\mu = \sum_{\lambda/\mu \text{ bbs length } r} (-t)^{\mathrm{ht}(\lambda/\mu)-\ell(\mu)} q^{\#\mathrm{cols}(\lambda/\mu)-\ell(\mu)} s_\lambda \qquad \begin{pmatrix} \textit{sum over broken border} \\ \textit{strips } \lambda/\mu \textit{ of length } r \end{pmatrix}$$

For $\lambda, \mu \in \mathbb{Y}_n$ let

$$K_{\lambda\mu} = \mathrm{Card}(B(\lambda)_\mu)$$

so that $K_{\lambda\mu}$ is the number of SSYT of shape λ and weight μ. For a partition λ let λ' be the conjugate parittion to λ.

Corollary 8.4. . *Let μ be a partition of n. Then*

$$h_\mu = \sum_{\lambda\in\mathbb{Y}_n} K_{\lambda\mu} s_\lambda, \qquad e_\mu = \sum_{\lambda\in\mathbb{Y}_n} K_{\lambda'\mu'} s_\lambda, \qquad p_\mu = \sum_{\lambda\in\mathbb{Y}_n} \chi^\lambda_{S_n}(\mu) s_\lambda,$$

and

$$\tilde{q}_\mu(q,1) = \sum_{\lambda\in\mathbb{Y}_n} \chi^\lambda_{H_n}(\mu) s_\lambda.$$

Proof of the Theorem.

Proof. Let $P \in B(\mu)$ and insert, by RSK column insertion,

$$P \leftarrow x_{i_1} \leftarrow \cdots \leftarrow x_{i_k}, \qquad \text{where} \quad i_1 \le \cdots \le i_r > i_{r+1} > \cdots > i_k.$$

□

Examples: For $n \in \{1,2,3,4\}$

$$(\chi^\lambda_{H_3}(\mu)) = MATRIX \qquad \text{and} \qquad (K_{\lambda\mu}) = MATRIX.$$

9 Week 6: Catalan algebraic combinatorics

1. (Generating function for Catalan) Show that if

$$G(x) = \sum_{n\in\mathbb{Z}_{\ge0}} C_n x^n \qquad \text{then} \qquad G(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

2. (binomial formula for Catalan). Show that

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

3. (recursion for Catalan). Show that $C_0 = 1$ and

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Sketch:

$$\frac{1}{x}(G(x)-1) = \sum_{n\in\mathbb{Z}_{\ge0}} C_{n+1}x^n = \sum_{n\in\mathbb{Z}_{ge0}} \Big(\sum_{i=0}^{n} C_i C_{n-i}\Big) x^n = G(x)^2.$$

Solve for $G(x)$ and use the binomial theorem to expand. The coefficient of x^n comes out to $\frac{1}{n+1}\binom{2n}{n}$.

4. (dimension of TL) Show that

$$C_n = \sum_{k=0}^{\lfloor n/2\rfloor} f^2_{(n-k,k)}.$$

5. (Chebyshev polynomials?)

6. Find a bijection between $NC(S_n)$ and noncrossing matchings of $\{1, \ldots, 2n\}$. Describe the resulting partial order on noncrossing matchings.

7. Let G_n be the lattice of subsets of $\{1, \ldots, n\}$, let $G_q(n)$ be the lattice of $\mathbb{F}_q$ subspaces of $\mathbb{F}_q^n$, and let $NC(n)$ be the lattice of noncrossing partitions. Show that the number of maximal chains is

$$\#\mathrm{fl}(G_n) = n!, \qquad \#\mathrm{fl}(G_q(n)) = [n]!, \qquad \#\mathrm{fl}(NC(n)) = (n+1)^{n-1}.$$

8. (dimension of Brauer) Show that

$$1 \cdot 3 \cdot 5 \cdots (2k-1) = \sum_{\lambda \in Y_k \cup Y_{k-1} \cup \cdots} b_\lambda^2$$

The *lattice of noncrossing partitions of* W is the interval of W (as a poset in the order determined by translation factoizations) given by

$$NC(W) = W_{[1,c]}, \qquad \text{where } c \text{ is a Coxeter element of } W.$$

The *cluster complex* is

$$\Upsilon(W) = ???.$$

The *algebraic parking space* is the Gordon module for the rational Cherednik algebra given by

$$\mathrm{Park}_W^{\mathrm{alg}}(m) = \mathrm{sgn} \otimes L_{m+\frac{1}{h}}(\mathrm{triv}).$$

The genus-g Hurwitz number is

$$H_g(\lambda) = \#\{\text{transitive factorizations of } \gamma_\lambda \text{ into reflections}\},$$

where γ_λ is a permutation of cycle type λ and transitive means that the group generated by the factors acts transitively on $\{1, \ldots, n\}$. THIS DEFINITION IS MISSING THE g ON THE RIGHT HAND SIDE.

Theorem 9.1. *The number of maximal chains in* $NC(W)$ *is*

$$\frac{1}{|W|} h^n n!.$$

10 Week 7: G/B for $GL_n(\mathbb{F}_q)$

10.1 Generators and relations for $GL_n(\mathbb{F}_q)$

Let E_{ij} be the $n \times n$ matrix with 1 in the (i,j)-entry and 0 elsewhere. For $i, j \in \{1, \ldots, n\}$ and $c \in \mathbb{F}$ and $d \in \mathbb{F}^\times$, define

$$x_{ij}(c) = 1 + cE_{ij}, \qquad \text{and} \qquad h_i(d) = 1 + (d-1)E_{ii}.$$

For $i \in \{1, \ldots, n-1\}$ and $c \in \mathbb{F}$, define

$$y_i(c) = 1 + (c-1)E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i}.$$

Identify each permutation $w \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ with the matrix

$$w = E_{1,w(1)} + E_{2,w(2)} + \cdots + E_{n,w(n)}.$$

Also use the notations

$$h(d_1,\ldots,d_n)=h_1(d_1)\cdots h_n(d_n),\qquad s_i=y_i(0)\qquad\text{and}\qquad s_{ij}=1-E_{ii}-E_{jj}+E_{ij}+E_{ji}.$$

The *x-interchange relations* are

$$\begin{array}{ll} x_{ij}(c_1)x_{ij}(c_2)=x_{ij}(c_1+c_2), & \\ x_{ij}(c_1)x_{ik}(c_2)=x_{ik}(c_2)x_{ij}(c_1), & x_{ik}(c_1)x_{jk}(c_2)=x_{jk}(c_2)x_{ik}(c_1), \\ x_{ij}(c_1)x_{jk}(c_2)=x_{jk}(c_2)x_{ij}(c_1)x_{ik}(c_1c_2), & x_{jk}(c_1)x_{ij}(c_2)=x_{ij}(c_2)x_{jk}(c_1)x_{ik}(-c_1c_2), \end{array}$$

where we assume that $i<j<k$. The hh-relations are

$$h_i(d)h_j(e)=h_j(e)h_i(d)\qquad\text{and}\qquad h(d_1,\ldots,d_n)h(e_1,\ldots,e_n)=h(d_1e_1,\ldots,d_ne_n).$$

The *h-past-x relation* is

$$h(d_1,\ldots,d_n)x_{ij}(c)=x_{ij}(cd_id_j^{-1})h(d_1,\ldots,d_n).\qquad\text{(GLhpastx)}$$

The *w-past-h and w-past-x relations* are

$$wh_i(c)=h_{w(i)}(c)w,\qquad wh(d_1,\ldots,d_n)=h(d_{w(1)},\ldots,d_{w(n)})w\qquad wx_{ij}(c)=x_{w(i)w(j)}(c)w.$$

The reflection relations and the building relations are the relations for rearranging ys. The *reflection relation* is

$$y_i(c_1)y_i(c_2)=\begin{cases} y_i(c_1+c_2^{-1})h_i(c_2)h_{i+1}(-c_2^{-1})x_{i,i+1}(c_2^{-1}), & \text{if } c_2\neq 0,\\ x_{i,i+1}(c_1), & \text{if } c_2=0.\end{cases}\qquad\text{(GLref)}$$

The *building relation* is

$$y_i(c_1)y_{i+1}(c_2)y_i(c_3)=y_{i+1}(c_3)y_i(c_1c_3+c_2)y_{i+1}(c_1).\qquad\text{(GLbldg)}$$

The *h-past-y relation* is (letting $h(d_1,\ldots,d_n)=h_1(d_1)\cdots h_n(d_n)$)

$$h(d_1,\ldots d_n)y_i(c)=y_i(cd_id_{i+1}^{-1})h(d_1,\ldots,d_{i-1},d_{i+1},d_i,d_{i+2},\ldots,d_n).\qquad\text{(GLhpasty)}$$

The *x-past-y relations* are

$$\begin{array}{ll} x_{i,i+1}(c_1)y_i(c_2)=y_i(c_1+c_2)x_{i,i+1}(0), & \\ x_{ik}(c_1)y_k(c_2)=y_k(c_2)x_{ik}(c_1c_2)x_{i,k+1}(c_1), & x_{i,k+1}(c_1)y_k(c_2)=y_k(c_2)x_{ik}(c_1), \\ x_{ij}(c_1)y_i(c_2)=y_i(c_2)x_{i+1,j}(c_1), & x_{i+1,j}(c_1)y_i(c_2)=y_i(c_2)x_{ij}(c_1)x_{i+1,j}(-c_1c_2), \end{array}\qquad\text{(GLxpasty)}$$

where $i<k$ and $i+1<j$.

10.1.1 The normal form algorithm

Let

$$N=\tfrac{1}{2}n(n-1).$$

Let $(i_1,\ldots,i_N)$ be the sequence

$$(i_1,\ldots,i_N)=(1,\ 2,1,\ 3,2,1,\ \ldots,\ n-1,n-2,\ldots,2,1)$$

Let $(\beta_1, \ldots, \beta_N)$ be the sequence

$$(\beta_1, \ldots, \beta_N) = \begin{pmatrix} \varepsilon_1 - \varepsilon_2, & \varepsilon_1 - \varepsilon_3, & \ldots, & \varepsilon_1 - \varepsilon_{n-1}, & \varepsilon_1 - \varepsilon_n, \\ & \varepsilon_2 - \varepsilon_3, & \ldots, & \varepsilon_2 - \varepsilon_{n-1}, & \varepsilon_2 - \varepsilon_n, \\ & & & \vdots & \vdots \\ & & & \varepsilon_{n-2} - \varepsilon_{n-1}, & \varepsilon_{n-2} - \varepsilon_n, \\ & & & & \varepsilon_{n-1} - \varepsilon_n \end{pmatrix}$$

For $c \in \mathbb{F}_q$ and $i \in \{1, \ldots, n-1\}$ define

$$y_i(c) = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad y_i(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For $a \in \mathbb{F}_q$ and $i, j \in \{1, \ldots, n\}$ with $i < j$ define

$$x_{\varepsilon_i - \varepsilon_j}(a) = x_{ij}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

Theorem 10.1. *Let $g \in GL_n(\mathbb{F})$. The normal form algorithm determines*

$$c_1, \ldots, c_N \in \mathbb{F} \cup \{\infty\} \qquad \textit{and} \qquad a_1, \ldots, a_N \in \mathbb{F} \qquad \textit{and} \qquad d_1, \ldots, d_n \in \mathbb{F}^\times$$

such that

$$g = y_{i_1}(c_1) \cdots y_{i_N}(c_N) h_1(d_1) \cdots h_n(d_n) x_{\beta_1}(a_1) \cdots x_{\beta_N}(a_N).$$

10.2 The Bruhat decomposition

The *flag variety* is

$$G/B = \{gB \mid g \in GL_n(\mathbb{F}_q)\}.$$

The *Bruhat decomposition* is the double coset decomposition

$$G/B = \bigsqcup_{w \in W} BwB.$$

10.3 The Bruhat order

Define

$$y_i(c) = ??? \qquad \text{and} \qquad y_i(\infty) = 1.$$

The *Schubert variety for w* is

$$\overline{BwB} = \{y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell)B \mid c_1, \ldots, c_\ell \in \mathbb{F}_q \cup \{\infty\}\}$$

The Bruhat order is the partial order on S_n defined by

$$\overline{BwB} = \bigsqcup_{v \le w} BvB.$$

The *partial flag variety* is

$$G/P = \{gP \mid g \in GL_n(\mathbb{F}_q)\}.$$

Then

$$G/P = \bigsqcup_{w\in W^P} BwP.$$

For $w \in W^P$, the *Schubert variety for* w is

$$\overline{BwP} = \{y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)P \mid c_1,\ldots,c_\ell \in \mathbb{F}_q \cup \{\infty\}\}$$

The Bruhat order is the partial order on W^P defined by

$$\overline{BwP} = \bigsqcup_{v\le w} BvP.$$

11 Week 8: Moment graphs and Kazhdan-Lusztig polynomials

11.1 Lecture 22: Moment graphs and $H_T(G/B)$

Let

$$L = \mathbb{Z}\text{-span}\{x_1,\ldots,x_n\} \qquad\text{and}\qquad S = \mathbb{C}[x_1,\ldots,x_n].$$

The **moment graph** G has vertices S_n and labeled edges

$$x \stackrel{x_i-x_j}{\longrightarrow} s_{ij}x, \qquad \text{if } x < s_{ij}x.$$

A *sheaf* $\mathcal{F}$ *on* G is a collection of

$$\text{an } S\text{-module } \mathcal{F}^x \text{ for each vertex } x\in V, \qquad \text{an } S\text{-module } \mathcal{F}^{(x,y)} \text{ for each edge } x\to y,$$

and S-module morphisms

$$\rho_x^{(x,y)}\colon \mathcal{F}^x \to \mathcal{F}^{(x,y)} \quad\text{and}\quad \rho_y^{(x,y)}\colon \mathcal{F}^y \to \mathcal{F}^{(x,y)} \qquad \text{for each edge } (x,y)\in E,$$

such that

$$\text{if } (x,y)\in E \quad\text{then}\quad l(x,y)\cdot\mathcal{F}^{(x,y)} = 0.$$

Let $\mathcal{T}$ be the topology on W generated by the sets

$$W_{\le x} = \{y \in W \mid y \le x\}, \qquad \text{for } x\in W.$$

The collection $\mathcal{T}$ is the smallest collection of subsets of W which contains all the W_x and is closed under unions and intersections. Let $\mathcal{F}$ be a sheaf on G and let $U\in\mathcal{T}$. A *section of* $\mathcal{F}$ over U is an element of

$$\mathcal{F}(U) = \Big\{(f_x)_{x\in U} \in \bigoplus_{x\in U}\mathcal{F}^x \;\Big|\; \text{if } x,y\in U \text{ and } (x,y)\in E \text{ then } \rho_x^{(x,y)}(f_x) = \rho_y^{(x,y)}(f_y)\Big\}.$$

A *sheaf morphism* from $\mathcal{F}_1$ to $\mathcal{F}_2$ is a collection of

$$\begin{aligned}&\text{an } S\text{-module morphism } \varphi^x\colon \mathcal{F}_1^x \to \mathcal{F}_2^x \text{ for each vertex } x\in V,\\ &\text{an } S\text{-module morphism } \varphi^{(x,y)}\colon \mathcal{F}_1^{(x,y)} \to \mathcal{F}_2^{(x,y)} \text{ for each edge } (x,y)\in E,\end{aligned}$$

such that if $(x,y)\in E$ then

$$\varphi^{(x,y)}\rho_x^{(x,y)} = \rho_x^{(x,y)}\varphi^x \quad\text{and}\quad \varphi^{(x,y)}\rho_y^{(x,y)} = \rho_y^{(x,y)}\varphi^y. \qquad \textit{PICTURE}$$

The *structure sheaf* of G has

$$\mathcal{F}^x = S, \qquad \text{for } x \in S_n, \qquad \text{and} \qquad \mathcal{F}^{(x,y)} = \frac{S}{l(x,y)S}, \qquad \text{for } (x,y) \in E,$$

and, for $(x,y) \in E$,

$$\begin{array}{rccc} \rho_x^{(x,y)}\colon & S & \to & \frac{S}{l(x,y)S} \\ & p & \mapsto & p + l(x,y)S \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \rho_y^{(x,y)}\colon & S & \to & \frac{S}{l(x,y)S} \\ & p & \mapsto & p + l(x,y)S \end{array}$$

The space $\mathcal{Z}(S_n)$ of (global) sections of the structure sheaf $\mathcal{Z}$ is an S-algebra with scalar mutliplication, addition and multiplication given componentwise: if $f, g \in \mathcal{Z}(S_n)$ and $p \in S$ and $x \in W$ then

$$(pf)_x = pf_x, \qquad (f+g)_x = f_x + g_x, \qquad (fg)_x = f_x g_x.$$

Theorem 11.1. *As S-algebras, $\mathcal{Z}(S_n) \cong H_T(G/B)$.*

11.2 Lecture 23: Sheaves on moment graphs

A graded free S-module is a graded S-module M such that there exist $r \in \mathbb{Z}_{>0}$ and $j_1, \ldots, j_r \in \mathbb{Z}$ such that

$$M \cong S[j_1] \oplus \cdots S[j_r], \qquad \text{as graded } S\text{-modules.}$$

The *graded rank* of M is

$$\mathrm{grk}(M) = q^{j_1} + \cdots q^{j_r}.$$

A *BMP sheaf* on G, or *Braden-MacPherson sheaf*, is a sheaf $\mathcal{B}$ on G such that

(BMP1) If $x \in W$ then $\mathcal{B}^x$ is a graded free S-module;

(BMP2) If $(x,y) \in E$ then

$$\mathrm{im}(\rho_y^{(x,y)}) = \mathcal{B}^{(x,y)} \qquad \text{and} \qquad \ker(\rho_y^{(x,y)}) = l(x,y)\mathcal{B}^y.$$

(BMP3) If $U \in \mathcal{T}$ then

$$\begin{array}{ccc} \mathcal{B}(W) & \to & \mathcal{B}(U) \\ (f_x)_{x \in W} & \mapsto & (f_x)_{x \in U} \end{array} \qquad \text{is surjective,}$$

(BMP4) If $w \in W$ then

$$\begin{array}{ccc} \mathcal{B}(W) & \to & \mathcal{B}^w \\ (f_x)_{x \in W} & \mapsto & f_w \end{array} \qquad \text{is surjective,}$$

Theorem 11.2. *If $w \in W$ then there is, up to isomorphism, a unique BMP sheaf $\mathcal{B}(w)$ such that*

(a) $\mathcal{B}(w)$ is indecomposable, and

(b) $\mathcal{B}(w)^w = S$ and $\mathcal{B}(w)^x = 0$ unless $x \le w$.

Theorem 11.3. *If $y, w \in W$ then*

$$P_{y,w} = \mathrm{grk}(\mathcal{B}(w)^y) \qquad \textit{is the KL-polynomial for the pair } y, w.$$

Theorem 11.4. *Let $\mathcal{B}$ be a BMP sheaf. Then there are $w_1, \ldots, w_r$ and $l_1, \ldots, l_r \in \mathbb{Z}$ such that*

$$\mathcal{B} \cong \mathcal{B}(w_1)[l_1] \oplus \cdots \oplus \mathcal{B}(w_r)[l_r].$$

Remark 11.5. The map

$$\begin{array}{ccc} (\text{BMP sheaves on } W) & \longleftrightarrow & (T\text{-equivariant perverse sheaves on } G/B) \\ \mathcal{B}(w) & \longmapsto & IC(\overline{BwB}) \end{array}$$

is an equivalence of categories.

11.3 Lecture 24: Kazhdan-Lusztig polynomials

Let $H = \mathbb{Z}[t, t^{-1}]\text{-span}\{T_w \mid w \in W\}$ and let $\overline{}\colon H \to H$ be a $\mathbb{Z}$-linear map such that

$$\overline{q} = q^{-1} \qquad \text{and} \qquad \overline{T_w} = T_w + \sum_{y<w} a_{yw} T_y.$$

(The a_{yw} are sometimes called R-polynomials.)

Proposition 11.6. *There exists a unique $C_w \in H$ such that*

$$C_w = T_w + \sum_{y<w} P_{y,w} T_w, \quad \textit{with} \quad P_{y,w} \in \mathbb{Z}[t], \qquad \textit{and} \qquad \overline{C_w} = C_w.$$

Remark 11.7. In the case that H is the Hecke algebra and

$$\overline{T_w} = T_{w^{-1}}^{-1}, \qquad \text{for } w \in W,$$

the a_{yw} are called R-polynomials and the $P_{y,w}$ are called *KL-polynomials*.

12 Week 9: Macdonald and Koornwinder polynomials

12.1 Macdonald polynomials

Fix $n \in \mathbb{Z}_{>0}$. The symmetric group S_n acts on $\mathbb{Z}^n$ by defining

$$s_i(\mu_1, \ldots, \mu_n) = (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_n), \qquad \text{for } i \in \{1, \ldots, n-1\}.$$

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$. The minimal length permutation $v_\mu \in S_n$ such that $v_\mu \mu$ is weakly increasing is given by

$$v_\mu(i) = 1 + \#\{i' \in \{1, \ldots, i-1\} \mid \mu_{i'} \le \mu_i\} + \#\{i' \in \{i+1, \ldots n\} \mid \mu_{i'} < \mu_i\}.$$

The symmetric group S_n acts on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{-1}]$ by permuting the variables,

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n), \qquad \text{for } i \in \{1, \ldots, n-1\}.$$

The polynomial ring $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm 1}]$ has $\mathbb{C}$-basis $\{x^\mu \mid \mu \in \mathbb{Z}^n\}$ and if $\mu \in \mathbb{Z}^n$ then $s_i x^\mu = x^{s_i \mu}$.
Define operators $\partial_i\colon \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm 1}] \to \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm 1}]$ by

$$\partial_i = (1 + s_i) \frac{1}{x_i - x_{i+1}}, \qquad \text{for } i \in \{1, \ldots, n-1\}.$$

The *electronic Macdonald polynomials* E_μ for $\mu \in \mathbb{Z}^n$ are determined by

(E0) $E_{(0,0,\ldots,0)} = 1$,

(E1) If $\mu_i > \mu_{i+1}$ then

$$E_{s_i \mu} = \Big(\partial_i x_i - t x_i \partial_i + \frac{(1-t) q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}} \Big) E_\mu,$$

where $v_\mu \in S_n$ is minimal length such that $v_\mu \mu$ is weakly increasing,

(E2) $E_{(\mu_n + 1, \mu_1, \ldots, \mu_{n-1})} = q^{\mu_n} x_n E_\mu(x_2, \ldots, x_n, q^{-1} x_1)$, (E3) If $k \in \mathbb{Z}$ then $E_{(\mu_1-k,\ldots,\mu_n-k)} = (x_1\cdots x_n)^{-k}E_\mu(x_1,x_2,\ldots,x_n)$.

Let $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \cdots \ge \lambda_n$. The *bosonic Macdonald polynomial* $P_\lambda = P_\lambda(q,t) = P_\lambda(x_1,\ldots,x_n;q,t)$ is

$$P_\lambda(q,t) = \frac{1}{W_\lambda(t)}\sum_{w\in S_n} w\Big(E_\lambda \prod_{i<j}\frac{x_i - tx_j}{x_i - x_j}\Big),$$

where $W_\lambda(t)$ is the appropriate constant which makes the coefficient of x^λ equal to 1 in $P_\lambda(q,t)$.

12.2 Koornwinder polynomials

Fix $n \in \mathbb{Z}_{>0}$. The group $W_{\rm fin}$ is generated by $s_1,\ldots,s_{n-1},s_n$ with relations

$$RELATIONS$$

The group $W_{\rm fin}$ acts on $\mathbb{Z}^n$ by defining

$$\begin{aligned} s_i(\mu_1,\ldots,\mu_n) &= (\mu_1,\ldots,\mu_{i+1},\mu_i,\ldots,\mu_n), \qquad \text{for } i\in\{1,\ldots,n-1\}, \text{ and}\\ s_n(\mu_1,\ldots,\mu_n) &= (\mu_1,\ldots,\mu_{n-1},-\mu_n), \end{aligned}$$

Let $\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{Z}^n$. The minimal length permutation $v_\mu \in S_n$ such that $v_\mu\mu$ is antidominant is given by

$$v_\mu(i)?? =??1 + \#\{i' \in \{1,\ldots,i-1\} \mid \mu_{i'} \le \mu_i\} + \#\{i' \in \{i+1,\ldots n\} \mid \mu_{i'} < \mu_i\}.$$

The group $W_{\rm fin}$ acts on $\mathbb{C}[x_1^{\pm1},\ldots,x_n^{-1}]$ by

$$\begin{aligned} (s_if)(x_1,\ldots,x_n) &= f(x_1,\ldots,x_{i+1},x_i,\ldots,x_n), \qquad \text{for } i\in\{1,\ldots,n-1\}, \text{ and}\\ (s_nf)(x_1,\ldots,x_n) &= f(x_1,\ldots,x_n^{-1}),\\ (s_0f)(x_1,\ldots,x_n) &= f(q^{?}??x_1,x_2,\ldots,x_n) \end{aligned}$$

Define operators $\partial_i\colon \mathbb{C}[x_1^{\pm},\ldots,x_n^{\pm1}] \to \mathbb{C}[x_1^{\pm},\ldots,x_n^{\pm1}]$ by

$$\partial_i = (1+s_i)\frac{1}{x_i - x_{i+1}}, \qquad \text{for } i \in \{1,\ldots,n-1\}.$$

The *electronic Macdonald polynomials* E_μ for $\mu \in \mathbb{Z}^n$ are determined by

(E0) $E_{(0,0,\ldots,0)} = 1$,

(E1) If $i \in \{1,\ldots,n-1\}$ and $\mu_i > \mu_{i+1}$ then

$$E_{s_i\mu} = \Big(\partial_i x_i - tx_i\partial_i + \frac{(1-t)q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)}}{1-q^{\mu_i-\mu_{i+1}}t^{v_\mu(i)-v_\mu(i+1)}}\Big)E_\mu,$$

where $v_\mu \in W_{\rm fin}$ is minimal length such that $v_\mu\mu$ is antidominanat,

(E2) $E_{(\mu_n+1,\mu_1,\ldots,\mu_{n-1})} = q^{\mu_n}x_nE_\mu(x_2,\ldots,x_n,q^{-1}x_1)$,

Let $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$. The *bosonic Koornwinder polynomial* $P_\lambda = P_\lambda(x_1,\ldots,x_n;q,t_0,t_n,u_0,u_n)$ is

$$P_\lambda = \frac{1}{W_\lambda(t)}\sum_{w\in S_n} w\Big(E_\lambda\Big(\prod_{i=1}^n\frac{(1-t_?u_?x_i)(1+t_?u_?x_i)}{1-x_i^2}\Big)\Big(\prod_{i<j}\frac{x_i-tx_j}{x_i-x_j}\Big)\Big),$$

where $W_\lambda(t)$ is the appropriate constant which makes the coefficient of x^λ equal to 1 in $P_\lambda(q,t)$.

13 Definitions of the symmetric functions

13.1 The power sum symmetric functions p_μ

Eefine p_r for $r \in \mathbb{Z}_{\ge 0}$ by

$$p_r = x_1^r + x_2^r + \cdots + x_n^r \qquad \text{and define} \qquad p_\nu = p_{\nu_1} p_{\nu_2} \cdots p_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers.

13.2 The elementary symmetric functions e_μ

Define e_r for $r \in \mathbb{Z}_{\ge 0}$ by

$$\sum_{r\in\mathbb{Z}_{\ge 0}} e_r z^r = \prod_{i=1}^n (1 + x_i z) \qquad \text{and define} \qquad e_\nu = e_{\nu_1} e_{\nu_2} \cdots e_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers.

13.3 The homogeneous symmetric functions h_μ

Define h_r for $r \in \mathbb{Z}_{\ge 0}$ by

$$\sum_{r\in\mathbb{Z}_{\ge 0}} h_r z^r = \prod_{i=1}^n \frac{1}{1 - x_i z} \qquad \text{and define} \qquad h_\nu = h_{\nu_1} h_{\nu_2} \cdots h_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers.

13.4 The little q's

Following [Mac, (Ch. III (2.10)], define q_r for $r \in \mathbb{Z}_{\ge 0}$ by

$$\sum_{r\in\mathbb{Z}_{\ge 0}} q_r z^r = \prod_{i=1}^n \frac{1 - t x_i z}{1 - x_i z} \qquad \text{and define} \qquad q_\nu = q_{\nu_1} q_{\nu_2} \cdots q_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers. In plethystic notation CHECK THIS

$$q_\nu = e_\nu[X(t-1)] \qquad \text{and} \qquad e_\nu = q_\nu\Big[\frac{X}{1-t}\Big].$$

13.5 The little g's

For a symbol a define the infinite product

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots .$$

Define g_r for $r \in \mathbb{Z}_{\ge 0}$ by

$$\sum_{r\in\mathbb{Z}_{\ge 0}} g_r z^r = \prod_{i=1}^n \frac{(t x_i z; q)_\infty}{(x_i z; q)_\infty} \qquad \text{and define} \qquad g_\nu = g_{\nu_1} g_{\nu_2} \cdots g_{\nu_\ell},$$

for a sequence $\nu = (\nu_1, \ldots, \nu_\ell)$ of positive integers.

13.6 The nonsymmetric Macdonald polynomials E_μ

Let $\mathbb{C}[X] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The symmetric group S_n acts on $\mathbb{C}[X]$ by permuting $x_1, \ldots, x_n$. Let

$$\mathbb{C}[X]^{S_n} = \{g \in \mathbb{C}[X] \mid \text{if } w \in S_n \text{ then } wg = g\} \qquad \text{the ring of symmetric functions.}$$

Letting $s_1, \ldots, s_{n-1}$ denote the simple transpositions in S_n,

$$(s_i f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n).$$

For $f \in \mathbb{C}[X]$ and $i \in \{1, \ldots, n-1\}$ define

$$\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}.$$

Let $\mathbb{Z}^n_{\geq 0}$ denote the set of length n sequences $\mu = (\mu_1, \ldots, \mu_n)$ of nonnegative integers (sometimes called the set of weak compositions). Define E_μ for $\mu \in \mathbb{Z}^n_{\geq 0}$ by setting $E_{(0,0,\ldots,0)} = 1$ and using the following recursions:

(1) If $\mu_i > \mu_{i+1}$ then $\qquad E_{s_i\mu} = \Big(\partial_i x_i - t x_i \partial_i + \dfrac{(1-t)q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}{1 - q^{\mu_i - \mu_{i+1}} t^{v_\mu(i) - v_\mu(i+1)}}\Big) E_\mu,$
where $v_\mu \in S_n$ is minimal length such that $v_\mu \mu$ is weakly increasing, and

(2) $E_{(\mu_n+1, \mu_1, \ldots, \mu_{n-1})} = q^{\mu_n} x_n E_\mu(x_2, \ldots, x_n, q^{-1}x_1)$.

Explicitly, the permutation $v_\mu \in S_n$ which is minimal length such that $v_\mu \mu$ is weakly increasing is given by

$$v_\mu(i) = 1 + \#\{i' \in \{1, \ldots, i-1\} \mid \mu_{i'} \leq \mu_i\} + \#\{i' \in \{i+1, \ldots n\} \mid \mu_{i'} < \mu_i\}.$$

13.7 The symmetric Macdonald polynomials P_λ

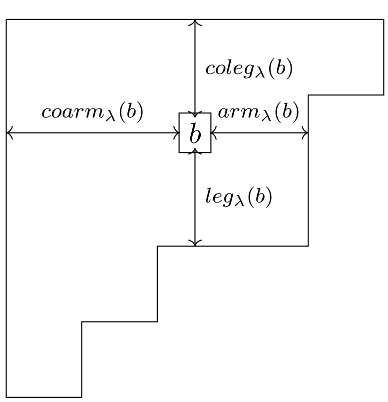
Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Define

$$P_\lambda(q,t) = \frac{1}{W_\lambda(t)} \sum_{w \in S_n} w\Big(E_\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j}\Big),$$

where $W_\lambda(t)$ is the appropriate constant which makes the coefficient of x^λ equal to 1 in $P_\lambda(q,t)$.

13.8 The big Js and the big Qs

Let λ be a partition and let λ' denote the conjugate partition to λ. Following, [Mac, VI (6.14)] for a box $b = (i,j)$ in λ define



$$\mathrm{coleg}_\lambda(b) = i - 1,$$

$$\mathrm{coarm}_\lambda(b) = j - 1, \qquad b = (i,j), \qquad \mathrm{arm}_\lambda(b) = \lambda_i - j,$$

$$\mathrm{leg}_\lambda(b) = \lambda'_j - i.$$

The *hook length* $h(b)$ and the *content* $c(b)$ of the box b are defined by

$$h(b) = \mathrm{arm}_\lambda(b) + \mathrm{leg}_\lambda(b) + 1 \qquad \text{and} \qquad c(b) = \mathrm{coarm}_\lambda(b) - \mathrm{coleg}_\lambda(b).$$

Define the *upper and lower hooks of a box* and the *upper and lower hook products of a partition* by

$$h^*_\lambda(b) = 1 - q^{\mathrm{arm}_\lambda(b)+1} t^{\mathrm{leg}_\lambda(b)}, \qquad h^\lambda_*(b) = 1 - q^{\mathrm{arm}_\lambda(b)} t^{\mathrm{leg}_\lambda(b)+1},$$

$$h^*_\lambda = \prod_{b\in\lambda} h^*_\lambda(b), \qquad h^\lambda_* = \prod_{b\in\lambda} h^\lambda_*(b),$$

The *integral form Macdonald polynomials* J_μ and the *dual Macdonald polynomials* Q_μ are given by [Mac, (8.3) and (8.11)]:

$$J_\mu(q,t) = h^\mu_* P_\mu(q,t) \qquad \text{and} \qquad Q_\mu(q,t) = \frac{h^\mu_*}{h^*_\mu} P_\mu(q,t).$$

13.9 The fermionic Macdonald polynomials $A_{\lambda+\delta}$

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\ge 0}$ with $\lambda_1 \ge \cdots \ge \lambda_n$ define

$$\lambda + \delta = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_{n-1} + 1, \lambda_n)$$

and

$$A_{\lambda+\delta}(q,t) = \Big(\prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\Big) \sum_{w\in S_n} (-1)^{\ell(w)} w E_{\lambda+\delta}.$$

Theorem 13.1. *(Weyl character formula for Macdonald polynomials)*

$$A_\delta(q,t) = \prod_{i<j}(x_i - tx_j) \qquad \textit{and} \qquad P_\lambda(q,qt) = \frac{A_{\lambda+\delta}(q,t)}{A_\delta(q,t)}.$$

13.10 The Schurs s_λ and the Big Schurs S_λ

The *Schur functions* s_λ and the *Big Schurs* S_λ are given in [Mac, Ch. I (7.7) and Ch. VI (8.9)] by the formulas

$$s_\lambda = \sum_\rho \frac{1}{z_\rho} \chi^\lambda_{S_n}(\rho) p_\rho \qquad \text{and} \qquad S_\lambda = S_\lambda(x;t) = \sum_\rho \frac{1}{z_\rho} \chi^\lambda_{S_n}(\rho) \Big(\prod_{i=1}^{\ell(\rho)} (1 - t^{\rho_i})\Big) p_\rho$$

where p_ρ is the power sum symmetric function and $\chi^\lambda_{S_n}$ are the irreducible characters of the symmetric group. In plethystic notation

$$S_\lambda = s_\lambda[X(1-t)] \qquad \text{and} \qquad s_\lambda = S_\lambda\Big[\frac{X}{1-t}\Big].$$

13.11 The modified Macdonald polynomials $\tilde{H}_\lambda(x;,q,t)$

Define $K_{\lambda\mu}(q,t)$ and the *modified Macdonald polynomials* $\widetilde{H}_\mu$ by the formulas

$$J_\mu = \sum_\mu K_{\lambda\mu}(q,t) S_\lambda \qquad \text{and} \qquad \widetilde{H}_\mu = \sum_\mu t^{n(\mu)} K_{\lambda\mu}(q,t^{-1}) s_\lambda. \qquad \text{(modMacdefn)}$$

In other words, change J_μ to $\widetilde{H}_\mu$ by changing S_λ to s_λ, changing t to t^{-1} and mutliplying by an overall factor of $t^{n(\mu)}$. This buries all the plethystic substitution into the switch from S_λ to s_λ. Write

$$\widetilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,t^{-1}) \qquad \text{so that} \qquad \widetilde{H}_\lambda(q,t;X) = \sum_\mu \widetilde{K}_{\lambda\mu}(q,t)s_\mu$$

The relation (modMacdefn) is not disssimilar to the relation

$$q_\mu = \sum_\lambda K_{\lambda\mu}S_\lambda. \qquad \text{and} \qquad h_\mu = \sum_\lambda K_{\lambda\mu}s_\lambda, \qquad \text{where} \quad K_{\lambda\mu} = K_{\lambda\mu}(0,1).$$

Remark 13.2. François Bergeron might define the *modified Macdonald polynomials* $\widetilde{H}_\mu = \widetilde{H}_\mu(q,t;x)$, using plethystic notation, by

$$\widetilde{H}_\mu(q,t;X) = t^{\eta(\mu)}P_\mu\Big(\frac{X}{1-t^{-1}};q,t\Big)\prod_{c\in\mu}(q^{a(c)} - t^{-(l(c)+1)}). \qquad \square$$

13.12 Transition matrices $\chi(t)$, $K(q,t)$, $Z(q,t)$, $\Psi(q,t)$ and $\mathcal{K}(q,t)$

Define $\chi_{\lambda\nu}(t)$ by

$$S_\lambda = \sum_\nu \chi_{\lambda\nu}(t)m_\nu.$$

Since $\chi_{\lambda\nu} = \langle S_\lambda(t), q_\nu(t)\rangle_{0,t}$ and $\langle q_\nu(t), m_\mu\rangle_{0,t} = \delta_{\nu\mu}$ and $\langle S_\lambda(t), s_\mu\rangle_{0,t} = \delta_{\lambda\mu}$ then

$$q_\nu(t) = \sum_\lambda \chi_{\lambda\nu}(t)s_\lambda,$$

Define $K_{\lambda\nu}(q,t)$ and $Z_{\lambda\mu}(q,t)$ by

$$J_\mu(q,t) = \sum_\lambda K_{\lambda\mu}(q,t)S_\lambda(t) \qquad \text{and} \qquad J_\lambda(q,t) = \sum_\mu Z_{\lambda\mu}(q,t)s_\mu.$$

Define $\Psi_{\mu\nu}(q,t)$ and $\mathcal{K}_{\lambda\mu}(q,t)$ by

$$J_\mu(q,t) = \sum_\mu \Psi_{\mu\nu}(q,t)m_\nu \qquad \text{and} \qquad J_\mu(q,qt) = \sum_\lambda \mathcal{K}_{\lambda\mu}(q,t)J_\lambda(q,t).$$

Remark 13.3. Relations: $\Psi(q,t) = Z(q,t)K(0,1)$ and $\Psi(q,t) = K(q,t)^t\chi(t)$. Since

$$s_\lambda = \sum_\mu K_{\lambda\mu}(0,1)m_\mu$$

then

$$\Psi_{\lambda\nu}(q,t) = \sum_\mu Z_{\lambda\mu}(q,t)K_{\mu\nu}(0,1), \qquad \text{and} \qquad \Psi_{\mu\nu}(q,t) = \sum_\lambda K_{\lambda\mu}(q,t)\chi_{\lambda\nu}(t). \qquad \square$$

Remark 13.4. A difference equation: $D_t\Psi = \mathcal{K}\Psi$ so that $\mathcal{K}$ is a connection matrix! Since

$$D_t\Psi = \Psi(q,qt) = \mathcal{K}(q,t)\Psi(q,t) = \mathcal{K}\Psi \qquad \text{and} \qquad D_tZ = Z(q,qt) = \mathcal{K}(q,t)Z(q,t) = \mathcal{K}Z,$$

then Ψ and Z a both solutions of the same difference equation, but with different initial conditions,

$$\Psi(q,q) = K(0,1) \qquad \text{and} \qquad Z(q,q) = \text{id}. \qquad \square$$

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