6.5 The interest sequence

Example. If you borrow \$500 on your credit card at 14% interest, find the amounts due at the end of two years if the interest is compounded

(a) annually,
(b) quarterly,
(c) monthly,
(d) daily,
(e) hourly,
(f) every second,
(g) every nanosecond,
(h) continuously.

(a) You owe

$$500 + 500(.14) = 500(1 + .14) \text{ after one year} \qquad \text{and} \qquad 500(1 + .14)(1 + .14) \text{ after two years.}$$

(b) You owe

$$500 + 500\Big(\frac{.14}{12}\Big) = 500\Big(1 + \frac{.14}{12}\Big) \quad \text{after one month.}$$

You owe

$$500\Big(1 + \frac{.14}{12}\Big)\Big(1 + \frac{.14}{12}\Big) \quad \text{after two months.}$$

You owe

$$500\Big(1 + \frac{.14}{12}\Big)^{24} \quad \text{after two years.}$$

(f) You owe

$$500 + 500\Big(\frac{.14}{365 \cdot 24 \cdot 3600}\Big) \quad \text{after one second.}$$

and

$$500\Big(1 + \frac{.14}{365 \cdot 24 \cdot 3600}\Big)^{2\cdot 365\cdot 24\cdot 3600} \qquad \text{after two years.}$$

In fact,

$$\begin{aligned}
\lim_{n\to\infty} 500\Big(1 + \frac{.14}{n}\Big)^{2n} &= 500 \lim_{n\to\infty} \Big(e^{\log\left(1+\frac{.14}{n}\right)}\Big)^{2n} \\
&= 500 \lim_{n\to\infty} e^{2n\log\left(1+\frac{.14}{n}\right)} \\
&= 500 \lim_{n\to\infty} e^{2\cdot .14\frac{\log(1+\frac{.14}{n})}{\frac{.14}{n}}} \\
&= 500 \lim_{n\to\infty} e^{.28\frac{\log(1+\frac{.14}{n})}{\frac{.14}{n}}} = 500e^{.28},
\end{aligned}$$

since

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

So you owe $500e^{.28}$ after two years if the interest is compounded continuously.

Note: $500(1 + .14)^2 = 649.80, \qquad 500\Big(1 + \frac{.14}{12}\Big)^{24} \approx 660.49, \quad \text{and} \quad 500e^{.28} \approx 661.58.$