

1.2 The complex numbers \mathbb{C}

The *complex numbers* is the \mathbb{R} -algebra

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\} \quad \text{with } i^2 = -1,$$

so that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) & \text{and} & & z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) & & & &= x_1 x_2 + i(x_1 y_2 + x_2 y_1) + i^2 y_1 y_2 \\ & & & & &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \end{aligned}$$

The *complex conjugation*, or *Galois automorphism*, is the \mathbb{R} -linear map

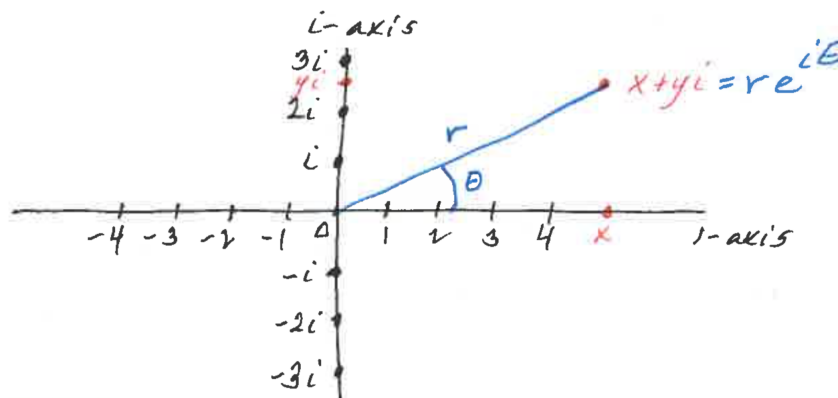
$$\bar{}: \mathbb{C} \rightarrow \mathbb{C} \quad \text{given by } \overline{x + iy} = x - iy.$$

The *norm*, or *length function*, on \mathbb{C} is the function

$$|\cdot|: \mathbb{C} \rightarrow \mathbb{R}_{\geq 0} \quad \text{given by } |x + iy| = \sqrt{x^2 + y^2}.$$

The *Hermitian form*, or *inner product*, on \mathbb{C} is

$$\langle \cdot, \cdot \rangle: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \quad \text{given by } \langle z_1, z_2 \rangle = z_1 \bar{z}_2.$$



Graphing complex numbers

If $r \in \mathbb{R}_{\geq 0}$ and $\theta \in \mathbb{R}$ then

$$r e^{i\theta} = r \cos \theta + i r \sin \theta.$$

If $z \in \mathbb{C}$ and $z \neq 0$ then

$$z^{-1} = \frac{1}{|z|^2} \bar{z},$$

since if $z = x + iy$ then

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{1}{(x + iy)(x - iy)} (x - iy) = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}.$$

HW: Show that if $z \in \mathbb{C}$ then $|z|^2 = z \bar{z}$.

HW: Show that if $z_1, z_2 \in \mathbb{C}$ then $|z_1 z_2| = |z_1| |z_2|$.