1.3 The binomial theorem

Let $k \in \mathbb{Z}_{\geq 0}$. Define k **factorial** by

$$0! = 1 \qquad \text{and} \qquad k! = k \cdot (k-1) \cdots 3 \cdot 2 \cdot 1 \text{ if } k \in \mathbb{Z}_{>0}.$$

Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$. Define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem 1.2. *Let $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$.*

(a) Let S be a set with cardinality n. Then
$\binom{n}{k}$ is the number of subsets of S with cardinality k.

(b) $\binom{n}{k}$ is the coefficient of $x^{n-k}y^k$ in $(x+y)^n$.

(c) If $k \in \{1, \ldots, n-1\}$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \qquad \textit{and} \qquad \binom{n}{0} = 1 \quad \textit{and} \quad \binom{n}{n} = 1.$$

This theorem says that the table of numbers

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
\ddots & & & & & \vdots & & & & & \ddots
\end{array}$$

are the numbers in **Pascal's triangle**

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\ddots & & & & & \vdots & & & & & \ddots
\end{array}$$

and that

$$\begin{array}{rcc}
(x+y)^0 & = & 1, \\
(x+y)^1 & = & x+y, \\
(x+y)^2 & = & x^2+2xy+y^2, \\
(x+y)^3 & = & x^3+3x^2y+3xy^2+y^3, \\
(x+y)^4 & = & x^4+4x^3y+6x^2y^2+4xy^3+y^4, \\
(x+y)^5 & = & x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5, \\
\vdots & & \vdots
\end{array}$$