1.5 Tutorial 3 MAST30005 Semester I, 2024: Categories

1. Carefully define the following
   (a) group
   (b) abeliangroup
   (c) ring
   (d) $\mathbb{Z}$-algebra
   (d) $\mathbb{F}$-algebra
   (d) R-algebra
   (e) commutativering
   (f) field

2. Carefully define the following
   (a) G-set
   (b) $\mathbb{Z}$-module
   (c) R-module
   (d) $\mathbb{F}$-module
   (e) $\mathbb{F}$-vector space

3. Carefully define the following
   (a) subgroup
   (b) subabeliangroup
   (c) subring
   (d) $\mathbb{Z}$-subalgebra
   (d) $\mathbb{F}$-subalgebra
   (d) R-subalgebra
   (e) subcommutativering
   (f) subfield

4. Carefully define the following
   (a) sub G-set
   (b) $\mathbb{Z}$-submodule
   (c) R-submodule
   (d) $\mathbb{F}$-submodule
   (e) $\mathbb{F}$-subspace

5. Carefully define the following
   (a) group morphism
   (b) abeliangroup morphism
   (c) ring morphism
   (d) $\mathbb{Z}$-algebra morphism
   (d) $\mathbb{F}$-algebra morphism
   (d) R-algebra morphism
   (e) commutativering morphism
   (f) field morphism

6. Carefully define the following

   (a) G-set morphism
   (b) $\mathbb{Z}$-module morphism
   (c) R-module morphism

(d) $\mathbb{F}$-module morphism
(e) $\mathbb{F}$-linear transformation

7. Carefully define the following
   (a) group isomorphism
   (b) abeliangroup isomorphism
   (c) ring isomorphism
   (d) $\mathbb{Z}$-algebra isomorphism
   (d) $\mathbb{F}$-algebra isomorphism
   (d) R-algebra isomorphism
   (e) commutativering isomorphism
   (f) field isomorphism

8. Carefully define the following
   (a) G-set isomorphism
   (b) $\mathbb{Z}$-module isomorphism
   (c) R-module isomorphism
   (d) $\mathbb{F}$-module isomorphism
   (e) $\mathbb{F}$-vector space isomorpihsm

9. Carefully define the following
   (a) group automorphism
   (b) abeliangroup automorphism
   (c) ring automorphism
   (d) $\mathbb{Z}$-algebra automorphism
   (d) $\mathbb{F}$-algebra automorphism
   (d) R-algebra automorphism
   (e) commutativering automorphism
   (f) field automorphism

10. Carefully define the following
   (a) G-set automorphism
   (b) $\mathbb{Z}$-module automorphism
   (c) R-module automorphism
   (d) $\mathbb{F}$-module automorphism
   (e) $\mathbb{F}$-vector space automorpihsm

11. Carefully define the following
   (a) kernel and image of a group morphism
   (b) kernel and image of an abeliangroup morphism
   (c) kernel and image of a ring morphism
   (d) kernel and image of a $\mathbb{Z}$-algebra morphism
   (d) kernel and image of a $\mathbb{F}$-algebra morphism
   (d) kernel and image of a R-algebra morphism
   (e) kernel and image of a commutativering morphism
   (f) kernel and image of a field morphism

12. (a) Let G be a group and let K be a subgroup of G. Show that

$$K \text{ is a normal subgroup of } G \qquad \text{if and only if} \qquad \begin{matrix} \text{there exists a group morphism} \\ \varphi\colon G \to H \text{ such that } \ker\varphi = K. \end{matrix}$$

(b) Let R be a ring and let I be a subabeliangroup of R. Show that

$$I \text{ is an ideal of } R \qquad \text{if and only if} \qquad \begin{matrix}\text{there exists a ring morphism} \\ \varphi\colon R \to S \text{ such that } \ker\varphi = I.\end{matrix}$$

(b) Let A be an R-algebra and let B be an R-submodule of A. Show that

$$B \text{ is an ideal of } A \qquad \text{if and only if} \qquad \begin{matrix}\text{there exists an } R\text{-algebra morphism} \\ \varphi\colon A \to C \text{ such that } \ker(\varphi) = B.\end{matrix}$$

(c) Let M be an R-module and let N be an R-submodule of M. Show that

$$N \text{ is an } R\text{-submodule of } M \qquad \text{if and only if} \qquad \begin{matrix}\text{there exists an } R\text{-module morphism} \\ \varphi\colon M \to P \quad \text{such that} \quad \ker(\varphi) = N.\end{matrix}$$

(d) Let V be an $\mathbb{F}$-vector space and let W be an $\mathbb{F}$-subspace of V. Show that

$$W \text{ is an } \mathbb{F}\text{-subspace of } V \qquad \text{if and only if} \qquad \begin{matrix}\text{there exists an } \mathbb{F}\text{-linear transformation} \\ \varphi\colon V \to P \quad \text{such that} \quad \ker(\varphi) = W.\end{matrix}$$

13. (a) Let $\varphi\colon G \to H$ be a group morphism. Show that

$$\frac{G}{\ker(\varphi)} \cong \mathrm{im}(\varphi) \qquad \text{as groups.}$$

(b) Let $\varphi\colon R \to S$ be a ring morphism. Show that

$$\frac{R}{\ker(\varphi)} \cong \mathrm{im}(\varphi) \qquad \text{as rings.}$$

(c) Let $\varphi\colon A \to B$ be an R-algebra morphism. Show that

$$\frac{A}{\ker(\varphi)} \cong \mathrm{im}(\varphi) \qquad \text{as } R\text{-algebras.}$$

(d) Let $\varphi\colon M \to N$ be an R-module morphism. Show that

$$\frac{M}{\ker(\varphi)} \cong \mathrm{im}(\varphi) \qquad \text{as } R\text{-modules.}$$

(e) Let $\varphi\colon V \to V$ be an $\mathbb{F}$-linear transformation. Show that

$$\frac{V}{\ker(\varphi)} \cong \mathrm{im}(\varphi) \qquad \text{as } \mathbb{F}\text{-vector spaces.}$$

14. (a) Let $\varphi\colon G \to H$ be a group morphism. Show that $\ker\varphi$ is a normal subgroup of G.
(b) Give an example of a group morphism $\varphi\colon G \to H$ such that $\mathrm{im}(\varphi)$ is a subgroup of H.
(c) Give an example of a group morphism $\varphi\colon G \to H$ such that $\mathrm{im}(\varphi)$ is not a subgroup of H.
(d) Let $\varphi\colon G \to H$ be a ring morphism. Explain how to use φ to make H into a G-set and show that $\mathrm{im}(\varphi)$ is an G-subset of H.

15. (a) Let $\varphi\colon R \to S$ be a ring morphism. Show that $\ker\varphi$ is an ideal of R.
(b) Give an example of a ring morphism $\varphi\colon R \to S$ such that $\mathrm{im}(\varphi)$ is an S-submodule of S.

   (c) Give an example of a ring morphism $\varphi\colon R \to S$ such that $\mathrm{im}(\varphi)$ is not an S-submodule S.
   (d) Let $\varphi\colon R \to S$ be a ring morphism. Explain how to use φ to make S into an R-module and show that $\mathrm{im}(\varphi)$ is an R-submodule of S.

16. Let R be a ring.
   (a) Let $\varphi\colon A \to B$ be an R-algebra morphism. Show that $\ker\varphi$ is an ideal of A.
   (b) Give an example of an R-algebra morphism $\varphi\colon A \to B$ such that $\mathrm{im}(\varphi)$ is a B-submodule of B.
   (c) Give an example of a R-algebra morphism $\varphi\colon A \to B$ such that $\mathrm{im}(\varphi)$ is not an B-submodule B.
   (d) Let $\varphi\colon A \to B$ be an R-algebra morphism. Explain how to use φ to make B into an A-module and show that $\mathrm{im}(\varphi)$ is an A-submodule of B.

17. Let $\mathbb{F}$ be a field. Show that an $\mathbb{F}$-vector space is the same thing as an $\mathbb{F}$-module.

18. Show that a ring is the same thing as a $\mathbb{Z}$-algebra.

19. Show that an abeliangroup is the same thing as a $\mathbb{Z}$-module.

20. Let R be a ring. Explain how R is an R-module. Show that an ideal of R is the same thing as an R-submodule of R.

21. Let A be an R-algebra. Explain how A is an A-module. Show that an ideal of A is the same thing as an A-submodule of A.

22. (a) Let G be a group. Show that a subgroup of G is the same as an injective group morphism $\varphi\colon H \to G$.
   (b) Let R be a ring. Show that a subring of R is the same as an injective ring morphism $\varphi\colon S \to R$.
   (c) Let A be an R-algebra. Show that an R-subalgebra A is the same as an injective R-algebra morphism $\varphi\colon C \to A$.
   (d) Let $\mathbb{K}$ be a field. Show that a subfield of $\mathbb{K}$ is the same as an injective field morphism $\varphi\colon \mathbb{F} \to \mathbb{K}$.
   (e) Let R be a ring and let M be an R-module. Show that an R-submodule of M is the same as an injective R-module morphism $\varphi\colon N \to M$.
   (f) Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Show that an $\mathbb{F}$-subspace of V is the same as an injective $\mathbb{F}$-linear transformation $\varphi\colon W \to V$.