19.7.6 Rings $\mathbb{Z}[\sqrt{d}]$ and $\mathbb{Z}[\frac{1}{2}+\frac{\sqrt{d}}{2}]$

225. Show that $\mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{-19}]$ is a PID that is not a Euclidean domain.

226. Let $\zeta = \frac{1+\sqrt{-3}}{2}$ and let $R=\mathbb{Z}[\zeta]$. Prove that the rings $R/(1+\zeta)R$ and $\mathbb{Z}/3\mathbb{Z}$ are isomorphic.

227. Let $\xi = (-1+\sqrt{-3})/2 \in \mathbb{C}$. Consider the **Eisenstein integers**
$$\mathbb{Z}[\xi] = \{a+b\xi \mid a,b\in\mathbb{Z}\}.$$
Show that $\mathbb{Z}[\xi]$ is a subring of $\mathbb{C}$. (Hint: $\xi^2+\xi+1=0$). Does there exist a homomorphism from $\mathbb{Z}[\xi]$ to $\mathbb{F}_2$?

228. What are the units in the following rings?

(a) $\mathbb{Z}$ (b) $\mathbb{Z}/5\mathbb{Z}$ (c) $\mathbb{Z}/15\mathbb{Z}$ (d) $\mathbb{Q}$

229. There are four rings (up to isomorphism) with four elements. Write down as much of the addition and multiplication tables of each of them as you can.

230. Find all the units in $\mathbb{Z}[\mathbf{i}] = \{a+b\mathbf{i} \mid a,b\in\mathbb{Z}\}$ (where $\mathbf{i}^2=-1$). (It might help to use the absolute value.)

231. Consider the ring
$$\mathbb{Z}[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b\in\mathbb{Z}\} \subseteq \mathbb{R}.$$
(a) Find a unit in $\mathbb{Z}[\sqrt{2}]$ other than ±1.
(b) Produce infinitely many units in $\mathbb{Z}[\sqrt{2}]$.

232. Pick your favourite ring amongst $\mathbb{Z}[e^{\frac{2\pi i}{3}}]$ and $\mathbb{Z}[\sqrt{-2}]$. Show that this ring is Euclidean with respect to the function $|z|^2$.

233. Let $d\in\mathbb{Z}$ be square-free and let $R=\mathbb{Z}[\sqrt{d}]$. Let $a\in R$ with $a\neq 0$ and $a\notin R^\times$. Show that a can be written as a product of irreducibles.

234. Show that $\mathbb{Z}[\sqrt{-7}]$ is not a unique factorization domain by using the identity $(1+\sqrt{-7})(1-\sqrt{-7}) = 2\cdot2\cdot2$ and the norm function $\phi\colon \mathbb{Z}[\sqrt{-7}]\to\mathbb{Z}$ given by $\phi(m+n\sqrt{-7}) = m^2+7n^2$.

235. Prove that $I = \{m+n\sqrt{-7} \mid m\in 7\mathbb{Z}\}$ is an ideal of $\mathbb{Z}[\sqrt{-7}]$ and determine whether I is a principal ideal.

236. Prove that $\mathbb{Z}[\sqrt{-7}]$ is not a PID.

237. The ring $R=\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain with size function $N\colon \mathbb{Z}[\sqrt{-2}]\to\mathbb{Z}_{\geq 0}$ given by
$$N(m+n\sqrt{-2}) = m^2+2n^2.$$
(a) Show that $1+\sqrt{-2}$ and $1+2\sqrt{-2}$ are irreducible in R.
(b) Prove that the units in R are 1 and -1.
(c) Prove that the ideal I generated by the two elements $2+2\sqrt{-2}$ and $4+3\sqrt{-2}$ is the principal ideal $\sqrt{-2}R$.

238. Let $R=\mathbb{Z}[\sqrt{-5}]$.

(a) Show that $2R$ is not prime and that $11R$ is prime.
(b) Let $I = 2R + (1+\sqrt{-5})R$ and $J = 2R + (1-\sqrt{-5})R$. Let IJ be the ideal of R generated by the set $\{i \cdot j \mid i \in I, j \in J\}$. Show that $2R = IJ$ and that I is prime.

239. Let $\eta = \frac{1}{2}(1+\sqrt{-19})$ and let $\mathbb{Z}]\eta] = \{x + y\eta \mid x, y \in \mathbb{Z}\}$.

(a) Determine $\mathbb{Z}[\eta]^\times$.

(b) Show that 2 and 3 are irreducible in $\mathbb{Z}[\eta]$.

(c) Let $N\colon \mathbb{C} \to \mathbb{R}$ be given by $N(z) = z\bar{z}$. Let I be an ideal of $\mathbb{Z}[\eta]$. Let $a \in I$ be such that $N(a)$ is minimal in $\{N(b) \mid b \in \mathbb{Z}[\eta] - \{0\}\}$. Show that

$$I = a\mathbb{Z}[\eta].$$

(d) Show that $\mathbb{Z}[\eta]$ is a PID and $\mathbb{Z}[\eta]$ is not a Euclidean domain.

240. Let $R = \mathbb{Z}[\sqrt{2}]$.

(a) Show that the ring R is a Euclidean domain with size function $N(a + b\sqrt{2}) = |a^2 + 2b^2|$.
(b) Compute $\gcd(7, -29 + 26\sqrt{2})$,
(c) Prove that there is an isomorphism

$$\frac{R}{(\sqrt{2}-3)R} \cong \mathbb{Z}/7\mathbb{Z}.$$