2.12 Proof of properties of primitive polynomials

Lemma 2.13. *Let R be a UFD. For each irreducible element $p \in R$ let*

$$\begin{matrix} R & \to & R/pR \\ c & \mapsto & \bar{c} = c + pR \end{matrix} \quad \text{and} \quad \begin{matrix} \pi_p\colon & R[x] & \to & \frac{R}{pR}[x] \\ c_0 + \cdots + c_k x^k & \mapsto & \overline{c_0} + \cdots + \overline{c_k} x^k \end{matrix}$$

be the quotient map and the corresponding homomorphism between polynomial rings.

Let $f(x) \in R[x]$. Then $f(x)$ is not primitive if and only if

$$\text{there exists an irreducible element } p \in R \text{ such that } \hat{\pi}_p\big(f(x)\big) = 0.$$

Proof.
$\Rightarrow$: Assume $f(x) = c_0 + c_1 x + \cdots + c_k x^k$ is not primitive.
Then there exists $p \in R$ irreducible such that p divides c_0, p divides c_1, …, p divides c_k.
So $c_0, c_1, \ldots, c_k \in pR$.
So $\pi_p(c_0) = \pi_p(c_1) = \cdots = \pi_p(c_k) = 0$.
So $\hat{\pi}_p\big(f(x)\big) = \pi_p(c_0) + \pi_p(c_1)x + \cdots + \pi_p(c_k)x^k = 0$.

$\Leftarrow$: Assume that $f(x) = c_0 + c_1 x + \cdots + c_k x^k$ and that there exists an irreducible element $p \in R$ such that $\hat{\pi}_p\big(f(x)\big) = 0$.
Then $\pi_p(c_0) = \pi_p(c_1) = \cdots = \pi_p(c_k) = 0$.
So $c_0, c_1, \ldots, c_k \in pR$.
So p divides c_0, p divides c_1, …, and p divides c_k.
So $f(x)$ is not primitive. □

Lemma 2.14. ***(Gauss' Lemma)*** *Let R be a UFD. Let $f(x), g(x) \in R[x]$ be primitive polynomials. Then $f(x)g(x)$ is a primitive polynomial.*

Proof. Proof by contrapositive:
To show: If $f(x)g(x)$ is not primitive then either $f(x)$ is not primitive or $g(x)$ is not primitive.
Assume $f(x)g(x)$ is not primitive.
Then, by Lemma 2.13, there exists an irreducible element $p \in R$ such that

$$\hat{\pi}_p\big(f(x)g(x)\big) = 0, \qquad \text{where} \qquad \hat{\pi}_p\colon R[x] \to \tfrac{R}{pR}[x]$$

is the homomorphism between polynomial rings induced by the quotient map $\pi_p\colon R \to R/pR$.
Since $\hat{\pi}_p$ is a homomorphism,

$$\hat{\pi}_p\big(f(x)g(x)\big) = \hat{\pi}_p\big(f(x)\big)\hat{\pi}_p\big(g(x)\big) = 0.$$

By Lemma 16.6, since p is irreducible then pR is a prime ideal.
Thus, by Theorem 4.47, R/pR and $\frac{R}{pR}[x]$ are integral domains.
So either

$$\hat{\pi}_p\big(f(x)\big) = 0 \quad \text{or} \quad \hat{\pi}_p\big(g(x)\big) = 0.$$

Thus, by Lemma 2.13,

either $f(x)$ is not primitive or $g(x)$ is not primitive.

□