16.2 Problem sheet: Rings

1. Let R be a commutative ring and let $x \in R$. Show that the ideal generated by x in R is equal to the set
$$Rx = \{rx \mid r \in R\}.$$

2. Let R be a factorial ring. Let $a_0, a_1, \ldots, a_n \in R$. A **greatest common divisor**, $\gcd(a_0, a_1, \ldots, a_n)$, of $a_0, a_1, \ldots, a_n$ is an element $d \in R$ such that
   (a) d divides a_i for all $i = 0, 1, \ldots, n$.
   (b) If d' divides a_i for all $i = 0, 1, \ldots, n$ then d' divides d.

3. Let R be a UFD and let $a_0, a_1, \ldots, a_n \in R$. Show that
   (a) $\gcd(a_0, a_1, \ldots, a_n)$ exists.
   (b) $\gcd(a_0, a_1, \ldots, a_n)$ is unique up to multiplication by a unit.

4. Let R be a UFD and let $p \in R$ be an irreducible element. Show that (p) is a prime ideal of R.

5. Show that the ring of integers $\mathbb{Z}$ with size function given by
$$\begin{array}{rccc} \sigma\colon & \mathbb{Z} - \{0\} & \to & \mathbb{Z}_{\geq 0} \\ & a & \mapsto & |a|. \end{array}$$
is a Euclidean domain.

6. Let $\mathbb{F}$ be a field. Show that $\mathbb{F}[x]$ with
$$\begin{array}{rccc} \sigma\colon & \mathbb{F}[x] - \{0\} & \to & \mathbb{Z}_{\geq 0} \\ & p(x) & \mapsto & \deg\big(p(x)\big) \end{array}$$
is a Euclidean domain.

7. Show that $\mathbb{Z}[x]$ with
$$\begin{array}{rccc} \sigma\colon & \mathbb{Z}[x] - \{0\} & \to & \mathbb{Z}_{\geq 0} \\ & p(x) & \mapsto & \deg\big(p(x)\big) \end{array}$$
is not a Euclidean domain.

8. Show that $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ with
$$\begin{array}{rccc} \sigma\colon & \mathbb{Z}[i] - \{0\} & \to & \mathbb{Z}_{\geq 0} \\ & a + bi & \mapsto & a^2 + b^2 \end{array}$$
is a Euclidean domain.

9. Show that $\mathbb{Z}[x]$ is a UFD that is not a principal ideal domain.

10. Show that $\mathbb{Z}[\sqrt{-5}]$ is a UFD that is not a PID.

11. Show that the ideal generated by 2 and x in $\mathbb{Z}[x]$ is not a principal ideal.

12. Show that $R = \big\{a + b(1 + \sqrt{19}i)/2 \mid a, b \in \mathbb{Z}\big\}$ is a principal ideal domain that is <u>not</u> a Euclidean domain.

13. (Eisenstein criterion) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ and let $p \in \mathbb{Z}_{>0}$ be a prime integer.
    Assume that
    (a) p does not divide a_n,
    (b) p divides each of $a_{n-1}, a_{n-2}, \ldots, a_0$,
    (c) p^2 does not divide a_0.

    Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$.
14. Let $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ and let p be a prime integer such that p does not divide a_n. Let $\hat{\pi}_p \colon \mathbb{Z}[x] \to \mathbb{Z}/p\mathbb{Z}[x]$ be the canonical homomorphism (see Ex. X). If $\hat{\pi}_p\big(f(x)\big)$ is irreducible in $\mathbb{Z}/p\mathbb{Z}[x]$ then $f(x)$ is irreducible in $\mathbb{Q}[x]$.
15. Show that if $f(x) \in \mathbb{Z}[x]$, $\deg\big(f(x)\big) > 0$, and $f(x)$ is irreducible in $\mathbb{Z}[x]$ then $f(x)$ is irreducible in $\mathbb{Q}[x]$.
16. Let $f(x) \in \mathbb{Z}[x]$. Show that $f(x)$ is irreducible in $\mathbb{Z}[x]$ if and only if

$$\begin{array}{ll} & \text{either } f(x) = \pm p, \text{ where } p \text{ is a prime integer,} \\ \text{or} & f(x) \text{ is a primitive polynomial and } f(x) \text{ is irreducible in } \mathbb{Q}[x]. \end{array}$$

17. Show that if A is. a commutative ring then ideals of A are the same as submodules of A.
18. Let $\mathbb{F}$ be a field. Show that $\mathbb{F}^\times = \mathbb{F} - \{0\}$ and $\mathbb{F}/\mathbb{F}^\times = \{0, 1\}$.
19. Show that $\mathbb{Z}^\times = \{0, 1\}$ and $\mathbb{Z}/\mathbb{Z}^\times = \mathbb{Z}_{\geq 0}$.
20. Show that $\mathbb{C}[x, y]$ is a UFD that is not a PID.
21. Show that $\mathbb{Z}[x]$ is a UFD that is not a PID.
22. Prove that if R is a PID then R is a UFD.
23. Show that if R is a PID and $p \in R$ then p is irreducible if and only if pR is a maximal ideal of R.
24. Let R be an integral domain and let $p \in R$. Show that p is a unit if and only if $pR = R$.
25. Let R be a integral domain and let $p, q \in R$. Show that p divides q if and only if $qR \subseteq pR$.
26. Let R be a integral domain and let $p, q \in R$. Show that p is a proper divisor of q if and only if $qR \subsetneq pR \subsetneq R$.
27. Let R be a integral domain and let $p, q \in R$. Show that p is an associate of q if and only if $pR = qR$.
28. Let R be a integral domain and let $p \in R$. Show that p is irreducible if and only if p satisfies the conditions
    (a) $pR \neq 0$ and $pR \neq R$, and
    (b) if $q \in R$ and $qR \supsetneq pR$ then $qR = R$.
29. Show that $\mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{-19}]$ is a PID that is not a Euclidean domain.

30. Let $\mathbb{A}$ be a commutative ring. Show that $\mathbb{A}$ satisfies the cancellation law if and only if $\mathbb{A}$ has no zero divisors.
31. Show that if $\mathbb{F}$ is a field then $\mathbb{F}$ satisfies the cancellation law.
32. Show that $\mathbb{Z}$ satisfies the cancellation law.
33. Show that zero divisors are nilpotent in $\mathbb{Z}/7^4\mathbb{Z} = \mathbb{Z}/2401\mathbb{Z}$.
34. Let $\zeta = \frac{1+\sqrt{-3}}{2}$ and let $R = \mathbb{Z}[\zeta]$. Prove that the rings $R/(1+\zeta)R$ and $\mathbb{Z}/3\mathbb{Z}$ are isomorphic.
35. Give an example of an integral domain R, together with three nonzero elements a, b and c, such that a and b both divide c, the product ab does not divide c, and a and b have no common factors apart from units.
36. Let R be a finite integral domain. Prove that R is a field.
37. Let R be a ring. Let $f \in R[x]$. Suppose that there exists a nonzero $g \in R[x]$ with $fg = 0$. Prove that there exists a nonzero $r \in R$ with $rf = 0$.
38. Let R be a ring. Let S be a subset of R with $1 \in S$ with the property that if $s, t \in S$, then $st \in S$. Show that the relation defined by
$$(x, s) \sim (y, t) \text{ if there exists } u \in S \text{ such that } (xt - ys)u = 0$$
is an equivalence relation on $R \times S$.
39. Let F be an infinite field.
    (a) Prove that if $f(x) \in F[x]$ is such that $f(a) = 0$ for all $a \in F$, then $f(x) = 0$.
    (b) Let $f(x, y) \in F[x, y]$ be a polynomial such that $f(a, a) = 0$ for all $a \in F$. Prove that $f(x, y)$ is divisible by $x - y$.
40. Let R be a ring such that $0 = 1$. Show that $R = \{0\}$.
41. Let R be a ring and let $a, b, c \in R$. Show that
    (a) if $a + b = a + c$ then $b = c$,
    (b) $a0 = 0 = 0a$,
    (c) $a(-b) = (-a)b = -(ab)$,
    (d) $(-a)(-b) = ab$.
42. Let R be a ring. Show that a unit of R cannot be a zero-divisor.
43. True or false?
    (a) Every field is also a ring.
    (b) Every ring has at least two elements.
    (c) The nonzero elements in a ring form a group under multiplication.
44. Let $\xi = (-1 + \sqrt{-3})/2 \in \mathbb{C}$. Consider the *Eisenstein integers*
$$\mathbb{Z}[\xi] = \{a + b\xi \mid a, b \in \mathbb{Z}\}.$$
Show that $\mathbb{Z}[\xi]$ is a subring of $\mathbb{C}$. (Hint: $\xi^2 + \xi + 1 = 0$). Does there exist a homomorphism from $\mathbb{Z}[\xi]$ to $\mathbb{F}_2$?
45. What are the units in the following rings?

(a) $\mathbb{Z}$ (b) $\mathbb{Z}/5\mathbb{Z}$ (c) $\mathbb{Z}/15\mathbb{Z}$ (d) $\mathbb{Q}$

46. There are four rings (up to isomorphism) with four elements. Write down as much of the addition and multiplication tables of each of them as you can.

47. Find all the units in $\mathbb{Z}[\mathbf{i}] = \{a+b\mathbf{i} \mid a, b \in \mathbb{Z}\}$ (where $\mathbf{i}^2 = -1$). (It might help to use the absolute value.)

48. Consider the ring
$$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{R}.$$
   (a) Find a unit in $\mathbb{Z}[\sqrt{2}]$ other than ± 1.
   (b) Produce infinitely many units in $\mathbb{Z}[\sqrt{2}]$.

49. Show that the units in the polynomial ring $F[x]$, where F is a field, are the elements of $F \setminus \{0\}$.

50. The *characteristic* of a ring is the smallest positive integer n such that $1 + 1 + \cdots + 1 = 0$, where there are n 1's added. If such an integer does not exist, we declare the characteristic to be zero. Let R be a ring of characteristic p, where p is a prime number. Show that the function $\phi : R \to R$ defined by $\phi(x) = x^p$ is a ring homomorphism. (*It is called the Frobenius map.*)

51. Give an example of a homomorphism whose image is not an ideal.

52. Let $\phi : R \to S$ be a ring homomorphism. Define a map $\Phi : R[x] \to S[x]$ by
$$\Phi(a_0 + a_1x + \cdots + a_nx^n) = \phi(a_0) + \phi(a_1)x + \cdots + \phi(a_n)x^n$$
Show that Φ is a ring homomorphism.

53. Let I be an ideal of a ring R. Show that if I contains a unit of R, then $I = R$.

54. Let R be the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that there exists $X \in \mathbb{R}$ with $f(x) = 0$ for all $|x| > X$. [Note X depends on f]. Define an addition on R by
$$(f + g)(x) = f(x) + g(x)$$
and a multiplication by
$$(fg)(x) = \int_{-\infty}^{\infty} f(t)g(x - t)\, dt.$$
Is R a ring?

55. (a) Let $f = x^2$ and $d = 2x + 1$. Find $q, r \in \mathbb{Q}[x]$ such that $f = qd + r$ and $\deg r < \deg d$.
   (b) Show that $\mathbb{Z}[x]$ is not a Euclidean domain with respect to the degree function. [Hint: Show that the q and r that arise when the division algorithm is performed in $\mathbb{Q}[x]$ are unique]

56. Show that in an integral domain, if $ab = ac$ then $a = 0$ or $b = c$.

57. Recall that in a principal ideal domain, if d is a greatest common divisor of a and b, then there exists r, s such that $d = ar + bs$ (this is essentially by definition, as $dR = aR + bR$). Use this result to show that if e divides both a and b, then e divides d.

58. Pick your favourite ring amongst $\mathbb{Z}[e^{\frac{2\pi i}{3}}]$ and $\mathbb{Z}[\sqrt{-2}]$. Show that this ring is Euclidean with respect to the function $|z|^2$.

59. Recall that the characteristic of a ring R is the smallest positive integer n such that the sum of n 1's is equal to $0 \in R$, if such an n exists; otherwise the characteristic is defined to be 0. (For example, the characteristic of $\mathbb{Z}/6\mathbb{Z}$ is 6; the characteristic of $\mathbb{Q}$ is 0.)

    (a) Let R be a ring of characteristic n. Show that the canonical homomorphism from $\mathbb{Z}$ to R has kernel $n\mathbb{Z}$ and that R contains a subring isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

    (b) Show that the characteristic of an integral domain is either zero or a prime number.

    (c) Conclude that every integral domain either contains a subring isomorphic to $\mathbb{Z}$, or contains a subring isomorphic to the finite field $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p.

60. A *prime field* is a field with no proper subfields. Show that a prime field is isomorphic to either $\mathbb{Q}$ or the finite field $\mathbb{F}_p$ for some prime p.

61. Let R be a ring. Show that the quotient ring $R[x]/(x-a)$ is isomorphic to R for any $a \in R$.

62. Show that the ideal $(x, y) \subset \mathbb{R}[x, y]$ is *not* principal.

63. Prove the *Second Isomorphism Theorem*: Let R be a ring, I an ideal of R and S a subring of R. Then

    (a) $S + I = \{s + a \mid s \in S, a \in I\}$ is a subring of R.

    (b) $S \cap I$ is an ideal of S.

    (c) $(S + I)/I \cong S/(S \cap I)$ (as rings).

64. Prove the *Third Isomorphism Theorem*: Let $I \subset J$ be ideals of a ring R. Then J/I is an ideal of R/I and $(R/I)/(J/I) \cong R/J$ (as rings).

65. **(T2)** Let R and S be rings. Is the map $r \mapsto (r, 0)$ from R to $R \times S$ a ring homomorphism? How about the *diagonal map* $r \mapsto (r, r)$ from R to $R \times R$?

66. If R and S are rings, their product $R \times S = \{(r, s) \mid r \in R, s \in S\}$ is a ring with $(r, s) + (r', s') = (r + r', s + s')$ and $(r, s)(r', s') = (rr', ss')$.

    (a) Write down the additive and multiplicative identities in $R \times S$.

    (b) Is $\mathbb{Z}/8\mathbb{Z}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (as rings)?

    (c) Is $\mathbb{Z}/6\mathbb{Z}$ isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (as rings)?

67. Let R and S be rings. Is the map $R \to R \times S$, $r \mapsto (r, 0)$ a ring homomorphism? What about the diagonal map $R \to R \times R$, $r \mapsto (r, r)$?

68. Let R be a ring. If I, J are ideals of R, the *sum* of I and J is defined by

$$I + J = \{x + y \mid x \in I, y \in J\} \subset R.$$

    (a) Show that $I + J$ is an ideal of R.

    (b) Prove the *Chinese Remainder Theorem*:
    If $I + J = R$ then $R/(I \cap J) \cong R/I \times R/J$ (as rings).

(c) The classical Chinese remainder theorem says that if m and n are coprime integers, then for any a, b, the system of equations $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$ has a unique solution modulo mn. Show how this follows from the result called the Chinese remainder theorem above.

69. Prove that two elements a and b of an integral domain R are associates if and only $aR = bR$. [This result is not true in more general rings, there is a counterexample if R the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and there are many other counterexamples]

70. Let R be a ring whose only units are ± 1, and which contains an element x such that $x^2+bx+c = 0$, where $b, c \in \mathbb{Z}$ with $b \equiv \pm 1 \pmod 6$ and $c \equiv 5 \pmod 6$. This exercise guides you through a proof that R is not a Euclidean domain.

    (a) Let f be a Euclidean function. Let $d \in R$, Show that the canonical function from $\{0\} \cup \{r \in R \mid f(r) < f(d)\}$ surjects onto R/dR.
    (b) Choose a non-unit d such that $f(d)$ is minimal amongst all non-units in R. Show that $|R/dR| \leq 3$ and hence that it is a finite field of order 2 or 3.
    (c) Show that there are no solutions to $x^2 + bx + c$ in a finite field of order 2 or 3 and therefore no homomorphisms from R to a finite field of order 2 or 3 to arrive at a contradiction.
    (d) Show that the conditions of this problem are satisfied for $\mathbb{Z}[\frac{1+\sqrt{-163}}{2}]$ (which is a PID, a nontrivial fact beyond the scope of this course).

71. Show that if $R[X]$ is an integral domain for which every ideal is principal, then R is a field.

72. If we regard the reals $\mathbb{R}$ as a subring of the complex numbers $\mathbb{C}$, we can extend the inclusion to a homomorphism $\phi : \mathbb{R}[X] \to \mathbb{C}$ by defining $\phi(X) = i \in \mathbb{C}$. Show that ϕ induces an isomorphism $\mathbb{R}[X]/(X^2 + 1) \cong \mathbb{C}$.

73. Show that every ideal in $\mathbb{Z}/12\mathbb{Z}$ is principal. Is $\mathbb{Z}/12\mathbb{Z}$ a PID?

74. Find a greatest common divisor in $\mathbb{Z}[i]$ of $-1 + 7i$ and $18 - i$.

75. Let F be a field and $f(x) \in F[x]$ a polynomial such that $f(a) \neq 0$ for all $a \in F$. Show that if f has degree at most 3, then $f(x)$ is irreducible.

76. Find all irreducible polynomials of degree at most 3 in $\mathbb{F}_2[x]$. Show that $1 + x + x^4$ is irreducible in $\mathbb{F}_2[x]$.

77. Let R be a ring and p an element of R. Prove that p is prime if and only if $R/(p)$ is an integral domain.

78. Let R be a ring. Prove that R is an integral domain if and only if $R[x]$ is an integral domain.

79. Let F be a field and let R be a subring of F.

    (a) Prove that R is an integral domain.
    (b) Prove that $\mathrm{Frac}(R)$ is (isomorphic to) a subfield of F.

80. Let R be a nonzero ring. An element $a \in R$ is said to be nilpotent if there exists some positive integer n such that $a^n = 0$. Let $a \in R$. Prove that if a is nilpotent then $1 + a$ is a unit.

81. Let I and J be ideals of a ring R.

(a) Prove that the set of finite sums $\sum x_i y_i$ ($x_i \in I$, $y_i \in J$) of products of elements of I and J, is an ideal. This ideal is called the product ideal and denoted by IJ.

(b) Suppose that $I + J = R$. Prove that $IJ = I \cap J$.

(c) Suppose that $I + J = R$. Prove that if $IJ = 0$, then R is isomorphic to the product ring $(R/I) \times (R/J)$.

(d) Suppose that $I + J = R$ and $IJ = 0$. Find the idempotents corresponding to the product decomposition in (c). That is, find an idempotent $e \in R$ and an idempotent $e' \in R$ such that R/I is isomorphic to eR and R/J is isomorphic to $e'R$ (as rings). (an *idempotent* is an element e satisfying $e^2 = e$.)

82. Let a and b be integers with $\gcd(a, b) = 1$ (in $\mathbb{Z}$). Prove that the greatest common divisor of a and b in $\mathbb{Z}[i]$ is also 1.

83. Let M be an R-module. Show that for all $r \in R$ and $m \in M$ we have

    (a) $0m = 0$

    (b) $r0 = 0$

    (c) $(-r)m = -(rm) = r(-m)$.

84. Let U be an open subset of $\mathbb{C}$ (or more generally a Riemann surface), let $\mathcal{O}(U)$ be the set of holomorphic functions on U, and let $\mathcal{M}(U)$ be the set of meromorphic functions on U. Define addition and multiplication pointwise.

    (a) Prove that $\mathcal{O}(U)$ is an integral domain if and only if U is connected.

    (b) Prove that $\mathcal{M}(U)$ is a field if (and only if) U is connected.

85. Let R be a ring and assume that $e_1, e_2 \in R$ satisfy

$$\text{if } a \in R \text{ then } e_1 a = a e_1 = a \text{ and } e_2 a = a e_2 = a.$$

    Show that $e_1 = e_2$.

86. Show that the identity in a ring R is unique.

87. Let R be a ring and $a, b \in R$. Show that $0a = a0 = 0$ and $a(-b) = (-a)b = -(ab)$ and $(-a)(-b) = ab$.

88. Let R be a ring. Show that if $1 = 0$ then R consists of a single element.

89. Let $\zeta = e^{2\pi i/5}$. Show that $\mathbb{Z}[\zeta] = \{a + b\zeta \mid a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$.

90. Let R be an integral domain and assume that R satisfies

$$\text{if } x \in R \text{ then } x^2 = x.$$

    Show that R has exactly two elements.

91. Find $\mathbb{Z}^\times$.

92. Find $(\mathbb{Z}/5ZZ)^\times$.

93. Find $\mathbb{Q}^\times$.

94. Find $(\mathbb{Z}\times\mathbb{Z})^\times$.

95. Find $(\mathbb{Z}/15\mathbb{Z})^\times$.

96. Find $\mathbb{R}[X]^\times$.

97. Show that every field is a ring.

98. Give an example of a ring that does not have an identity.

99. Show that every ring has at least two elements.

100. Show that the nonzero elements in a field form a group under multiplication.

101. Show that addition in a ring is always commutative.

102. Give the multiplication table for $(\mathbb{Z}/12\mathbb{Z})^\times$. Identity this group.

103. Determine $\mathbb{Z}[i]^\times$.

104. Determine $\mathbb{Z}[\sqrt{2}]^\times$, where $\mathbb{Z}[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b\in\mathbb{Z}\}$.

105. Show that if R is an integral domain then $R[x]$ is an integral domain.

106. Give and example of a ring R and a subset $I\subseteq R$ such that I is a left ideal but not a right ideal.

107. Let R be a ring and let S be a non-empty subset of R. Prove that S is a subring of R if and only if S satistifes
$$\text{if } a,b\in S \text{ then } a-b\in S \text{ and } ab\in S.$$

108. Let R be a ring and let I be an ideal of R. Show that if I contains a unit from R then $I=R$.

109. Show that a field F has only two ideals.

110. Let R be a commutativering. Show that if R has exactly two ideals then R is a field.

111. Give an example of a ring R and $S\subseteq R$ such that S is additively and multiplicatively closed but S is not a subring of R.

112. Is $\mathbb{Z}$ isomorphic to $2\mathbb{Z}$ as rings without identity? Is $\mathbb{Z}$ isomorphic to $2\mathbb{Z}$ as abeliangroups?

113. Give an example of a ring homomorphism $\varphi\colon R\to S$ such that $\mathrm{im}(\varphi)$ is not an ideal in S.

114. Let R be a ring and let I be a subgroup of R. Prove that I is an ideal of R if and only if the operations on R/I are well defined.

115. Let R and S be rings. Prove that the map
$$\begin{array}{rccc} \varphi\colon & R & \to & R\oplus S \\ & r & \mapsto & (r,0) \end{array} \qquad \text{is a ring homomorphism.}$$

116. Let R be a ring. Prove that the map
$$\begin{array}{rccc} \varphi\colon & R & \to & R\oplus R \\ & r & \mapsto & (r,r) \end{array} \qquad \text{is a ring homomorphism.}$$

117. Prove that $\mathbb{Z}/8\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ as rings.

118. Prove that $\mathbb{Z}/15\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ as rings.

119. Let I, j be ideals of a ring R. Show that $I + J = \{x + y \mid x, y \in I\}$ is an ideal of R.

120. Let R be a ring and let I, J be ideals of R. Show that if $I + J = R$ then $R/(I \cap J) \cong R/I \oplus R/J$.

121. Let $m, n \in \mathbb{Z}_{>0}$. Prove that $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(m, n) = 1$.

122. Let R be a ring. Let I be an ideal of R and let S be a subring of R. Prove that
$$\frac{(S+I)}{I} \cong \frac{S}{S \cap I}.$$

123. Let R be a ring and let I, J be ideals in R with $I \subseteq J$. Prove that
$$\frac{R/I}{J/I} \cong R/J.$$

124. Let $\varphi\colon R \to S$ be a ring homomorphism. Prove that there is a bijection between YIKES YIKES YIKES.

125. Let R be a integral domain. Show that the characteristic of R is 0 or prime.

126. Let R be a ring with characteristic n. Show that R contains a subring isomorphic to $\mathbb{Z}/\mathbb{Z}$.

127. Let F be a field with no proper subfeilds. Show that $F = \mathbb{Q}$ or $\mathbb{F} = \mathbb{F}_p$.

128. Let R be a commutativering of characteristic p. Show that
$$\text{if } x, y \in R \text{ and } n \in \mathbb{Z}_{\geq 0} \quad \text{then} \quad (x+y)^{p^n} = x^{p^n} + y^{p^n}.$$

129. Let R be a commutativering of characteristic p. Show that
$$\begin{array}{rccc} F\colon & R & \to & R \\ & x & \mapsto & x^p \end{array} \qquad \text{is a ring homomorphism.}$$

130. Let R be an integral domain. Show that there exists a field F such that R is a subring of F.

131. Let $\varphi_1\colon R \to S_1$ and $\varphi_2\colon R \to S_2$ be ring homomorphisms. Show that
$$\begin{array}{rccc} \varphi\colon & R & \to & S_1 \oplus S_2 \\ & a & \mapsto & (\varphi_1(a), \varphi_2(a)) \end{array} \qquad \text{is a ring homomorphism.}$$

132. Let $\varphi\colon R \to S$ be a ring homomrphism. Show that the maps given by
$$\begin{array}{rccc} \varphi\colon & R[x] & \to & S[x] \\ & a_0 + a_1x + \cdots + a_nx^n & \mapsto & \varphi(a_0) + \varphi(a_1)x + \cdots + \varphi(a_n)x^n \end{array}$$
is a ring homomorphism.

133. Let $\mathbb{F}$ be a field. Determine $\mathbb{F}^\times$.

134. Let $\mathbb{F}$ be a field. Determine $\mathbb{F}[x]^\times$.

135. Let R be a commutativering and let $a \in R$. Show that

$$\begin{array}{ccc} R[x] & \to & R[x] \\ a_0 + a_1 x + \cdots + a_n x^n & \mapsto & a_0 + a_1(x-a)x + \cdots + a_n(x-a)^n \end{array} \qquad \text{is a ring isomorphism.}$$

136. Is $\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$ invertible in the ring $M_2(\mathbb{Z}/3\mathbb{Z})$?

137. Is $\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$ invertible in the ring $M_2(\mathbb{Z}/6\mathbb{Z})$?

138. Is $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ invertible in the ring $M_2(\mathbb{Z})$?

139. Is $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ invertible in the ring $M_2(\mathbb{Q})$?

140. Is $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ invertible in the ring $M_2(\mathbb{Z})$?

141. Is $\begin{pmatrix} X & 2 \\ 0 & 1 \end{pmatrix}$ invertible in the ring $M_2(\mathbb{R}[x])$?

142. Is $\begin{pmatrix} 1 & x^2+1 \\ 0 & 2 \end{pmatrix}$ invertible in the ring $M_2(\mathbb{R}[x])$?

143. Let $R = \mathbb{R}[X, Y]$ Show that $I = \mathbb{R}\text{-span}\{X, Y\}$ is an ideal of R that is not principal.

144. Show that every ideal in $\mathbb{Z}/12\mathbb{Z}$ is prinicpal and that $\mathbb{Z}/12\mathbb{Z}$ is not a PID.

145. Let D be a integral domain. Let $p, q \in D$ and assume that q divides p. Show that if p is a unit then q is a unit.

146. Let D be a integral domain. Let $p, q \in D$ and assume that q divides p. Show that if p is irreducible then either q is a unit or p and q are associates.

147. Let D be a integral domain. Let $p, q \in D$ and assume that q divides p and that p and q are associates. Show that p is irreducible if and only if q is irreducible.

148. Let $d \in ZZ$ be square-free and let $R = \mathbb{Z}[\sqrt{d}]$. Let $a \in R$ with $a \neq 0$ and $a \notin R^\times$. Show that a can be written as a product of irreducibles.

149. Let R be a UFD and let $a \in R$. Show that if a is irreducible then aR is prime.

150. Let R be a integral domain such that every element $a \in R$ that is nonzero and not a unit can be written as a product of irreducibles $a = p_1 p_2 \cdots p_n$. Show that R is.a UFD if and only if every irreducible element of R is prime.

151. Let $R = \mathbb{Z}/6\mathbb{Z}$. Find an element $a \in R$ such that aR is maximal and a is not irreducible.

152. Give an example of an integral domain R and an element $a \in R$ such that a is irreducible and aR is not maximal.

153. Let R be a PID and let S be an integral domain and let $\varphi\colon R \to S$ be a ring homomorphism. Show that either φ is an isomorphism or S is a field.

154. Let I, J and P be ideals in R with P prime. Show that if $IJ]subseteqP$ then either $I \subseteq P$ or $J \subseteq P$.

155. Determine the maximal ideals in $\mathbb{R}$.

156. Determine the maximal ideals in $\mathbb{Z}$.

157. Determine the maximal ideals in $\mathbb{Z}/11\mathbb{Z}$.

158. Determine the maximal ideals in $\mathbb{Z}/12\mathbb{Z}$.

159. Let $R = \mathbb{Z}[\sqrt{-5}]$. Show that $2R$ is not prime and that $11R$ is prime.

160. Let $R = \mathbb{Z}[\sqrt{-5}]$ and let $I = 2R + (1+\sqrt{-5})R$ and $J = 2R + (1-\sqrt{-5})R$. Let IJ be the ideal of R generated by the set $\{i, j \mid i \in I, j \in J\}$. Show that $2R = IJ$ and that I is prime.

161. Let $f = x^4 - 3x^3 + 2x^2 + 4x - 1$ and $g = x^2 - 2x + 3$ in $\mathbb{Z}/5\mathbb{Z}[x]$. Use long division to find $q, r \in \mathbb{Z}/5\mathbb{Z}[x]$ such that $\deg(r) < \deg(g)$ and $f = qg + r$.

162. Show that $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$ and not irreducible in $\mathbb{R}[x]$.

163. Show that $x^3 + 3x + 2$ is irreducible in $\mathbb{Z}/5\mathbb{Z}[x]$.

164. Let R be a commutativering. Show that if $a, b \in R$ then $R[x]/(x-a) \cong R[x]/(x-b)$.

165. Show that $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.

166. Let R be an integral domain and let $f, g \in R[x]$ and that g is monic. Show that there exist $q, r \in R[x]$ such that $f = gq + r$ and either $r = 0$ or $\deg(r) < \deg(g)$.

167. Let R be a integral domain. Show that if $R[x]$ is a PID then R is a field.

168. Determine whether $\mathbb{Q}[x]/\langle x^2 - 5x + 6\rangle$ is a field. Determine whether $\mathbb{Q}[x]/\langle x^2 - 6x + 6\rangle$ is a field.

169. Let $\frac{r}{s} \in \mathbb{Q}$ be. a reduced fraction and assume that $\frac{r}{s}$ is a root of $f = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$. Show that r divides a_0 and s divides a_n. Deduce that if f is monic and has a rational root then it has a root that is an integer that divides a_0.

170. Determine the maximal ideals in $\mathbb{R}[x]/x^2\mathbb{R}[x]$.

171. Determine the maximal ideals in $\mathbb{R}[x]/(x^2+1)\mathbb{R}[x]$.

172. Determine the maximal ideals in $\mathbb{C}[x]/(x^2+1)\mathbb{C}[x]$.

173. Show that if R is a PID then R satisfies ACC.

174. Show that if R is a commutativering for which all ideals are finitely generated then R satisfies ACC.

175. Show that if R is a commutativering which satisfies ACC then all ideals are finitely generated.

176. Let R be a PID. Show that

$$pR \text{ is prime} \Rightarrow p \text{ is irreducible in } R \Rightarrow pR \text{ is maximal} \Rightarrow R/pR \text{ is a field}$$
$$\Rightarrow R/pR \text{ is an integral domain} \Rightarrow pR \text{ is prime}$$

177. Factor 5, 7, and $4+3i$ into irreducibles in $\mathbb{Z}[i]$.

178. Let R be a UFD. A polynomial $a_0 + a_1x + \cdots + a_nx^n \in R[x]$ is primitive if it is not a constant and $a_0R + a_R + \cdots + a_nR = R$. Show that if $f \in R[x]$ is non-constant then there exist $a \in R$ and primitive polynomial $g \in R[x]$ such that $f = ag$ and that a and g are unique up to associates.

179. Show that every field is a PID.

180. Show that every field is a UFD.

181. Show that every PID is a UFD.

182. Show that every UFD is a PID.

183. Show that if R is a. UFD and $a, b \in R$ are irreducible then $aR = bR$.

184. Show that if R is a PID then $R[x]$ is a PID.

185. Show that if R is a UFD then $R[x]$ is a UFD.

186. Show that if R is a integral domain and $a \in R$ is irreducible then aR is prime.

187. Let R be a UFD. Let $a, p \in R$ and assume that p is irreducible. Show that if p divides a then p appears in every factorization of a.

188. Express $18x^2 - 12x + 48 \in \mathbb{Z}[x]$ as the product of a constant polynomial and a primitive polynomial.

189. Express $18x^2 - 12x + 48 \in \mathbb{Q}[x]$ as the product of a constant polynomial and a primitive polynomial.

190. Express $2x^2 - 3x + 6 \in \mathbb{Z}/7\mathbb{Z}[x]$ as the product of a constant polynomial and a primitive polynomial.

191. Factor $4x^2 - 4x + 8$ into a product of irreducibles in $\mathbb{Z}[x]$, $\mathbb{Q}[x]$ and $\mathbb{Z}/11\mathbb{Z}[x]$.

192. Prove that if r is a PID and $a, b \in R$ then any gcd of a, b can be written as an R-linear combination of a, b.

193. Let R be a PID and let S be an integral domain containing R. Let $a, b, d \in R$. Show that if $d = \gcd(a, b)$ in R then $d = \gcd(a, b)$ in S.

194. Let $p, q \in \mathbb{Z}$ be relatively prime. Show that they are relatively prime in $\mathbb{Z}[x]$.

195. Let R be a UFD and let $a, b, d \in R$. Show that $\gcd(da, db) = d\gcd(a, b)$.

196. Let R be a integral domain and let $a, b, d, d' \in R$. Show that if $dR = d'R$ and $d = \gcd(a, b)$ then $d' = \gcd(a, b)$.

197. Let R be an integral domain and let $a, b, q, r, \in R$. Show that if $a = qb + r$ then $d = \gcd(a, b)$ if and only if $d = \gcd(b, r)$.

198. Let φ be the ring morphism from $\mathbb{Z}[x]$ to $\mathbb{R}$ which is the identity on $\mathbb{Z}$ and takes x wot $1+\sqrt{2}$. Show that the kernel of φ is a principal ideal and find a generator for this ideal.

199. Show that $\mathbb{Z}[x]/\langle 2x-1\rangle \cong \mathbb{Z}[\frac{1}{2}]$, where $\mathbb{Z}[\frac{1}{2}]$ denotes the smallest subring of $\mathbb{Q}$ that contains $\mathbb{Z}$ and $\frac{1}{2}$.

200. Let $\mathbb{F}$ be a field. Let $f \in \mathbb{F}[x]$ have degree 2 or 3. Show that f is irreducible if and only if it has no roots in $\mathbb{F}$.

201. Determine whether x^2-12 is irreducible in $\mathbb{Q}[x]$.

202. Determine whether $8x^3+6x^2-9x+24$ is irreducible in $\mathbb{Q}[x]$.

203. Determine whether $2x^10-25x^3+10x^2-30$ is irreducible in $\mathbb{Q}[x]$.

204. Determine whether x^4-16x^2+4 is irreducible in $\mathbb{Q}[x]$.

205. Determine whether x^4-32x^2+4 is irreducible in $\mathbb{Q}[x]$.

206. Determine whether $x^4-x^3-x^2-x-2$ is irreducible in $\mathbb{Q}[x]$.

207. Determine whether $2x^4-5x^3+3x^2+4x-6$ is irreducible in $\mathbb{Q}[x]$.

208. Determine whether $7x^3+6x^2+4x+4$ is irreducible in $\mathbb{Q}[x]$.

209. Determine whether $9x^4+4x^3-x+7$ is irreducible in $\mathbb{Q}[x]$.

210. Determine whether $x^5-4x+22$ is irreducible in $\mathbb{Q}[x]$.

211. Determine whether $2x^5+12x^4-15x^3+18x^2-45x+3$ is irreducible in $\mathbb{Q}[x]$.

212. Determine whether x^4+1 is irreducible in $\mathbb{Q}[x]$.

213. Determine whether $x^2+2345x+125$ is irreducible in $\mathbb{Q}[x]$.

214. Determine whether $x^3+5x^2+10x+5$ is irreducible in $\mathbb{Q}[x]$.

215. Factor x^5+5x+5 into irreducible factors in $\mathbb{Q}[x]$ and in $\mathbb{F}_2[x]$.

216. Facto x^3+x^2+1 in $\mathbb{F}_2[x]$ and $\mathbb{F}_3[x]$.

217. Let $n \in \mathbb{Z}_{\geq 1}$. Show that there are infinitely many monic irreducible polynomials of degree n in $\mathbb{Q}[x]$.

218. List all monic polynoimals of degree ≤ 2 in $\mathbb{F}_3[x]$. Determine which of these polynomials are irreducible.

219. Determine all irreducible polynomials of degree ≤ 4 in $\mathbb{F}_2[x]$.

220. Let $\mathbb{F}$ be a field. Show that $\mathbb{F}$ is a Euclidean domain.

221. Show that the Eisenstein integers is a Euclidean domain.

222. Let $\eta = \frac{1}{2}(1+\sqrt{-19})$ and let $\mathbb{Z}]\eta] = \{x+y\eta \mid x,y \in \mathbb{Z}\}$. Determine $\mathbb{Z}[\eta]^\times$.

223. Let $\eta = \frac{1}{2}(1+\sqrt{-19})$ and let $\mathbb{Z}]\eta] = \{x+y\eta \mid x,y \in \mathbb{Z}\}$. Show that 2 and 3 are irreducible in $\mathbb{Z}[\eta]^\times$.

224. Let $\eta = \frac{1}{2}(1+\sqrt{-19})$ and let $\mathbb{Z}]\eta] = \{x + y\eta \mid x, y \in \mathbb{Z}\}$. Let $N\colon \mathbb{C} \to \mathbb{R}$ be given by $N(z) = z\bar{z}$. Let I be an ideal of $\mathbb{Z}[\eta]$. Let $a \in I$ be such that $N(a)$ is minimal in $\{N(b) \mid b \in \mathbb{Z}[\eta] - \{0\}\}$. Show that $I = a\mathbb{Z}[\eta]$.

225. Let $\eta = \frac{1}{2}(1+\sqrt{-19})$ and let $\mathbb{Z}]\eta] = \{x + y\eta \mid x, y \in \mathbb{Z}\}$. Show that $\mathbb{Z}[\eta]$ is a PID and $\mathbb{Z}[\eta]$ is not a Euclidean domain.

226. Let $R = \mathbb{Q}[X]$ and let $f = x^3 - 6x^2 + x + 4$ and $g = x^5 - 6x + 1$. Find a monic polynomial $d \in \mathbb{Q}[x]$ such that $d\mathbb{Q}[x] = f\mathbb{Q}[x] + g\mathbb{Q}[x]$.

227. Let $R = \mathbb{Q}[X]$ and let $f = x^3 - 6x^2 + x + 4$ and $g = x^4 - 6x^3 + 5$. Find a monic polynomial $d \in \mathbb{Q}[x]$ such that $d\mathbb{Q}[x] = f\mathbb{Q}[x] + g\mathbb{Q}[x]$.

228. Let $R = \mathbb{Q}[X]$ and let $f = x^3 + 2x^2 + 4x - 7$ and $g = x^2 + x - 2$. Find a monic polynomial $d \in \mathbb{Q}[x]$ such that $d\mathbb{Q}[x] = f\mathbb{Q}[x] + g\mathbb{Q}[x]$.

229. Show that
$$GL_2(\mathbb{Z}) \text{ is generated by } \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

230. Let R be a ring and let M be an R-module. Show that $0m = 0$.

231. Give the definition of a submodule of an R-module.

232. Let M be a simple R-module. Let $\varphi\colon M \to N$ be an R-module morphism. Prove that either φ is injective or $\varphi(m) = 0$ for all $m \in M$.

233. Give an example of a ring R and a monic polynomial $f(x) \in R[x]$ such that the number of roots of $f(x)$ in R is greater than the degree of f.

234. Give an example of a ring with exaclty 3 ideals.

235. Give an example of a finite abelian group that is not a direct sum of cyclic groups.

236. Give an example of a field with 6 elements.

237. Let $R = \mathbb{Z}[\sqrt{2}]$. The ring R is a Euclidean domain with size function $N(a + b\sqrt{2}) = |a^2 + 2b^2|$. Compute $\gcd(7, -29 + 26\sqrt{2})$ and prove that there is an isomorphism
$$\frac{R}{(\sqrt{2} - 3)R} \cong \mathbb{Z}/7\mathbb{Z}.$$

238. Let p be a prime such that $p = 1 \bmod 4$. Show that $x^2 + 1 = 0$ has a solution in $\mathbb{F}_p$.

239. Let p be a prime such that $p = 1 \bmod 4$. By factoring $x^2 + 1$ in $\mathbb{Z}[i]$ show that p is not a prime element of $\mathbb{Z}[i]$.

240. Let p be a prime such that $p = 1 \bmod 4$. Show that p has a factorization $p = \alpha\beta$ where $|\alpha| = |\beta| = \sqrt{p}$.

241. Let R be a ring. Give the definition of an R-module.

242. Let R be a ring and let M be an R-module. Prove that if $r \in R$ then $r0 = 0$.

243. Let R be a ring and let M be an R-module. Let $r \in R$ and $m \in M$. Assume that $m \neq 0$ and $rm = 0$. Prove that r is not a unit in R.

244. Give an example of a ring R, an R-module M and elements $r \in R$ and $m \in M$ with $m \neq 0$ and $rm = 0$.

245. Give an example of a ring which is not an integral domain.

246. Give an example of an ideal in a ring which is not a principal ldeal.

247. Give an example of a field F and a nonconstant polynomial $f \in F[x]$ such that $f'(x) = 0$.

248. Give an example of two different subfields of $\mathbb{C}$ that are isomorphic to each other.

249. Let $\zeta = e^{2\pi i/3}$. Define a ring homomorphism $\varphi\colon \mathbb{Z}[\zeta] \to \mathbb{Z}/7\mathbb{Z}$. Prove that your function φ is a ring homomrphism.

250. Determine the kernel of the homomorphism you constructed in the previous part. Give your answer in the form of an ideal generated by the minimum possible number of generators this ideal can have.

251. Let R and S be rings and let $\phi\colon R \to S$ be a ring homomorphism. Let J be an ideal of R and let $I = \{r \in R \mid \phi(r) \in J\}$.
    (a) Prove that I is an ideal in R.
    (b) Prove that if J is a prime ideal in S then I is a prime ideal in R.

252. Let R be a UFD. Let a and b be two elements of R with $\gcd(a, b) = 1$. Suppose that there exists $x \in R$ with $ab = x^2$. Prove that there exists $u, v, y, z \in R$ with u and v units such that $a = uy^2$ and $b = vz^2$.

253. Give a detailed step-by-step construction of the field $\mathbb{F}_8$ as the quotient of a polynomial algebra.

254. Give a detailed step-by-step construction of the field $\mathbb{F}_4$ as the quotient of a polynomial algebra.

255. Prove that there is no field homomorphism from $\mathbb{F}_4$ to $\mathbb{F}_8$.

256. Give the multiplication table for $\mathbb{F}_4$.

257. Give the multiplication table for $\mathbb{F}_8$.

258. Give the definition of a ring.

259. Give an example of a subset of a ring that is a subgroup under addition but is not a subring.

260. Give an example of a finite non-commutative ring.

261. Prove that a ring R is commutative if and only if R satisfies
$$\text{if } a, b \in R \quad \text{then} \quad a^2 - b^2 = (a + b)(1 - b).$$

262. Let R be a ring. What does ti mean to say that a in R is a zero divisor? Give the definition of what is means to say that R is a integral domain.

263. Give an example of a commutative ring without zero divisors that is not an integral domain.

264. Find a multiplicative inverse of $2x+1$ in $\mathbb{Z}/4\mathbb{Z}[x]$ and prove that it is unique.

265. Prove that every finite integral domain is a field.

266. Let R and S be rings and $\varphi\colon R \to S$ a ring homomorphism such that $\mathrm{im}(\varphi) \neq \{0\}$.

    (a) Show that if R has a multiplicative identity and S has no zero divisors then $\varphi(1)$ is a multiplicative identity for S.

    (b) Give an example to show that the conclusion of (a) may fail when S has zero divisors.

267. Let R be a commutative ring and let $a, b, \in R$. Give the definition of a greatest common divisor of a and b.

268. Let R be a commutative ring and let $a, b, \in R$. Give an example to show that greatest common divisor need not exist.

269. Let R be a PID and let $a, b \in R$ and let $d = \gcd(a, b)$. Prove that $aR + bR = dR$.

270. List all the elements of $\mathbb{F}_2[x]$ that are irreducible, monic and have degree at most three.

271. Show that $x^7 + x + 1$ is irreducible in $\mathbb{F}_2[x]$.

272. Decide whether $2x^5 + 6x^3 - 6$ is irreducible in $\mathbb{Q}[x]$.

273. Decide whether $x^7 - 2x^5 + 5x - 7$ is irreducible in $\mathbb{Z}[x]$.

274. Decide whether $x^4 - 3x - 2$ is irreducible in $\mathbb{Q}[x]$.

275. Decide whether $x^5 - 2x^4 + 6x + 2$ is irreducible in $\mathbb{R}[x]$.

276. Let R be a commutative unital ring. Give the definition of an ideal in R.

277. Give an example of a commutative unital ring and an ideal I of R such that I is not principal.

278. Let R be a commutative unital ring. Give the definition of a zero-divisor in R.

279. Let R be a commutative unital ring. An ideal I of R is primary if I satisfies:

    $$\text{if } a, b, \in R \text{ and } ab \in I \text{ and } b \notin I \text{ then there exists } n \in \mathbb{Z}_{>0} \text{ such that } a^n \in I.$$

    Prove that I is primary if and only if every zero-divisor in R/I is nilpotent.

280. Let R be a ring whose additive group $(R, +)$ is cyclic. Show that R is a commutative ring.

281. Give an example of unital rings R and S be unital rings and a ring homomorhpism $\varphi\colon R \to S$ such that $\varphi(1) \notin \{0, 1\}$.

282. Let R and S be unital rings and let $\varphi\colon R \to S$ be a ring homomorphism. Show that if S is an integral domain then $\varphi(1) \in \{0, 1\}$.

283. Let R and S be unital rings and let $\varphi\colon R \to S$ be a ring homomorphism. Show that if S is a division ring and φ is not the zero map then φ is injective.

284. Factor $2x^3 + 4x + 1 \in \mathbb{F}_5[x]$ as a product of irreducibles.

285. Factor $x^4 - 4 \in \mathbb{Q}[x]$ as a product of irreducibles.

286. Factor $x^4 - 4 \in \mathbb{R}[x]$ as a product of irreducibles.

287. Show that $2x^6 + 15x^3 - 10X + 30 \in \mathbb{Z}[x]$ is irreducible.

288. Show that $2x^4 + 2x^2 - 2X + 2 \in \mathbb{Q}[x]$ is irreducible.

289. Show that $x^4 - 10x^2 + 1 \in \mathbb{Q}[x]$ is irreducible.

290. Write $x + 2 \in \mathbb{Z}/6\mathbb{Z}[x]$ as a product of two linear polynomials in $\mathbb{Z}/6\mathbb{Z}[x]$.

291. Let R be a ring and $S \subseteq R$ a subset. Give the definition of a *subring of R generated by S*.

292. Let $T \subseteq M_2(\mathbb{F}_2)$ be given by

$$T = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

   (a) Show that T is a subring of the ring $M_2(\mathbb{F}_2)$.
   (b) Find an element $a \in T$ such that T is generated by the set $\{a\}$.

293. (a) Give the definition of what it means to say that an integral domain R is a UFD.
   (b) Give an example of an ID that is not a UFD.

294. Give an example of a finite unital commutative ring in which the group of units is not cyclic.

295. Define the *characteristic* of a unital ring, and show that finite IDs have prime characteristic.

296. (a) Give the definition of a *homomorphism* between rings.
   (b) Show that the rings $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$ are not isomorphic.

297. (i) Give the definition of an *ideal* in a commutative ring.
   (ii) Let R and S be commutative rings. Suppose that $\varphi\colon R \to S$ is a homomorphism and that $I \subseteq J$ are ideals in R. Show that $\varphi(I)$ is an ideal in $\varphi(J)$.

298. (a) Let R be an integral domain. Define what it means to say that an element $p \in R$ is prime.
   (b) Let R be a principal ideal domain (PID) and $p \in R$ a non-zero non-unit element. Show that if p is prime then pR is a maximal ideal in R.
   (c) Let $\mathbb{F}$ be a field and $f \in \mathbb{F}[X]$. Suppose that $a \in E \supseteq \mathbb{F}$ is root of f and that f is irreducible. Show that f divides every polynomial in $\mathbb{F}[X]$ that has a as a root.

299. Give an example of a polynomial in $\mathbb{F}_2[X]$ that is reducible but has no roots in $\mathbb{F}_2$.

300. Factorize the polynomial $X^5 + 2X^4 + 4X^3 + 4X^2 + 3X + 2$ into irreducible factors in $\mathbb{F}_5[X]$.

301. Determine whether $X^4 - 5X^2 - 30X - 15$ is irreducible in $\mathbb{Q}[X]$.

302. Determine whether $X^4 - 5X^2 - 30X - 15$ is irreducible in $\mathbb{Z}[X]$.

303. Determine whether $X^4 - X^2 - 2$ is irreducible in $\mathbb{Q}[X]$.

304. Determine whether $X^4 - X^2 - 2$ is irreducible in $\mathbb{Z}[X]$.

305. Determine whether $2X^2 - 4$ is irreducible in $\mathbb{Q}[X]$.

306. Determine whether $2X^2 - 4$ is irreducible in $\mathbb{Z}[X]$.

307. Let R be a ring. Define what it means to say that R is an integral domain.

308. Let R be a ring. Define what it means to say that R is a unique factorization domain.

309. Give an example of an integral domain that is not a Unique factorization doamin.

310. Show that if R is a UFD and $a \in R$ is irreducible then a is prime.

311. Give the definition of a ring homomrphism.

312. Let $f\colon R \to S$ be a ring homomorphism. Show that if R has an identity element 1_R then $f(1_R)$ is an identity element for the image of f.

313. Give an example of a ring homomorphism $f\colon R \to S$ between two unital rings which is nonzero and satisfies $f(1_R) \neq f(1_S)$.

314. Let R be a commutative ring. Give the definition of an ideal in R.

315. Let R be a commutative ring and let $S \subseteq R$ be a subring of R and let I and J be ideals of R. Show that if $S \subseteq I \cup J$ then $S \subseteq I$ or $S \subseteq J$.

316. Let I be an ideal of R. Show that $(R/I)[X] \cong R[X]/I[X]$.

317. Let $\mathbb{F} = \mathbb{Z}/3\mathbb{Z}$ and let I be the ideal of $\mathbb{F}[X]$ which is generated by the polynomials $f = X^4 + X^3 + X + 2$ and $g = X^4 + 2X^3 + 2X + 2$. Determine $\gcd(f, g)$.

318. Let $\mathbb{F} = \mathbb{Z}/3\mathbb{Z}$ and let I be the ideal of $\mathbb{F}[X]$ which is generated by the polynomials $f = X^4 + X^3 + X + 2$ and $g = X^4 + 2X^3 + 2X + 2$. Show that $X^4 + 2$ is an element of I.

319. Let $\mathbb{F} = \mathbb{Z}/3\mathbb{Z}$ and let I be the ideal of $\mathbb{F}[X]$ which is generated by the polynomials $f = X^4 + X^3 + X + 2$ and $g = X^4 + 2X^3 + 2X + 2$. Show that $\mathbb{E} = \mathbb{F}[X]/I$ is a field.

320. Let $\mathbb{F} = \mathbb{Z}/3\mathbb{Z}$ and let I be the ideal of $\mathbb{F}[X]$ which is generated by the polynomials $f = X^4 + X^3 + X + 2$ and $g = X^4 + 2X^3 + 2X + 2$. Find the prime factorizations of $X^4 + 2$ in $\mathbb{F}[X]$ and in $\mathbb{E}[X]$.

321. Determine whether the polynomial $2X^3 - 10X^2 + 50X + 10$ is irreducible in the ring $\mathbb{Z}[X]$.

322. Determine whether the polynomial $2X^3 - 10X^2 + 50X + 10$ is irreducible in the ring $\mathbb{Q}[X]$.

323. Determine whether the polynomial $2X^3 - 10X^2 + 50X + 10$ is irreducible in the ring $\mathbb{R}[X]$.

324. Determine whether the polynomial $2X^3 - 10X^2 + 50X + 10$ is irreducible in the ring $\mathbb{C}[X]$.

325. Decompose $X^4 + 1$ as a product of irreducibles in $\mathbb{Z}/5\mathbb{Z}[X]$.

326. Decompose $X^5 + X$ as a product of irreducibles in $\mathbb{Z}/2\mathbb{Z}[X]$.

327. Decompose $X^5 + 4X^4 - 3X^3 + X^2 + 7X + 11$ as a product of irreducibles in $\mathbb{Q}[X]$.

328. Let R be a ring. Define what it means to say that $e \in R$ is a multiplicative identity in R.

329. Let R be a ring. Show that R has at most one mutliplicative identity.

330. Let R be.a finite ring. Suppose that there is an element $a \in R - \{0\}$ that is not a zero-divisor. Show that R has a multiplicative identity.

331. Give the definition of a ring homomorphism.

332. Show that if there is an injective ring homomorphism $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ then m divides n.

333. Show that the rings $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$ are not isomorphic.

334. Determine whether $2x^5 + 6x^3 + 12x^2 + 6$ is irreducible in $\mathbb{Z}[X]$.

335. Determine whether $2x^5 + 6x^3 + 12x^2 + 6$ is irreducible in $\mathbb{Q}[X]$.

336. Determine whether $2x^5 + 6x^3 + 12x^2 + 6$ is irreducible in $\mathbb{R}[X]$.

337. Determine whether $X^4 + X + 1$ is irreducible in $\mathbb{Q}[X]$.

338. Give the definition of a principal ideal domain.

339. Show that $\mathbb{R}[X, Y]$ is not a principal ideal domain.

340. Let R be a pricipal ideal domain and $a \in R$. Show that if a is irreducible then the ideal generated by a is maximal.

341. Give the definition of a ring.

342. Let R be a ring. Prove that it $a \in R$ then $0a = 0$.

343. Give the definitions of integral domain and principal ideal domain.

344. Give an example of a ring that is an integral domain but not a principal ideal domain.

345. Let R be an integral domain. What does it mean to say that $a \in R$ is prime?

346. Let R be an integral domain. What does it mean to say that $a \in R$ is irreducible?

347. Let R be an integral domain. Prove that all prime elements are irreducible.

348. Give an example of an integral domain R in which not all irreducible elements are prime.

349. Give the definition of a Euclidean domain.

350. Prove that if F is a field then $F[X]$ is a Euclidean domain.

351. Use the Euclidean algorithm to find the greatest common divisor of the two elements $X^5 + X^4 + X^3 - 2X^2 - 2X - 2$ and $X^5 - 3X^3 - 2X^2 + 6$ in $\mathbb{Q}[X]$.

352. State Eisenstien's irreduciblity criterion for polynomials.

353. Determine whether $X^2 + 2X + 2$ is irreducible in $\mathbb{Q}[X]$.

354. Determine whether $X^{10} + 8X^5 - 2X$ is irreducible in $\mathbb{Q}[X]$.

355. Determine whether $X^{10} + 12X^3 - 24X + 6$ is irreducible in $\mathbb{Q}[X]$.

356. Determine whether $X^4 + 15X^3 + 7$ is irreducible in $\mathbb{Q}[X]$.

357. Give the definitions of Euclidean domain and unique factorization domain.

358. Give an example of a Euclidean domain that is not a UFD.

359. Give and example of a unique factorization domain that is not a Euclidean domain.

360. Suppose that R is an integral domain and let $a, b \in R$ be any two elements. Show that $(a) = (b)$ if and only if there is a unit $u \in R$ such that $a = ub$.

361. Use the Euclidean algorithm to find the gcd in $\mathbb{Q}[x]$ of $f_1(x) = x^3 - x^2 + x - 3$ and $f_2(x) = x^4 - x^3 + 3x^2 + x - 4$ and write it as a linear combination of f_1 and f_2.

362. Determine whether $x^4 - 20x^2 - 30x + 15$ is irreducible in $\mathbb{Q}[x]$.

363. Determine whether $x^4 - 20x^2 - 30x + 15$ is irreducible in $\mathbb{Z}[x]$.

364. Determine whether $x^4 - x^2 - 2$ is irreducible in $\mathbb{Q}[x]$.

365. Determine whether $x^4 - x^2 - 2$ is irreducible in $\mathbb{Z}[x]$.

366. Determine whether $4x^3 - 2$ is irreducible in $\mathbb{Q}[x]$.

367. Determine whether $4x^3 - 2$ is irreducible in $\mathbb{Z}[x]$.

368. Determine whether $x^4 + 2x^3 + 2x^2 + 3x + 1$ is irreducible in $\mathbb{Q}[x]$.

369. Determine whether $x^4 + 2x^3 + 2x^2 + 3x + 1$ is irreducible in $\mathbb{Z}[x]$.

370. Give the definitions of integral domain and field.

371. Give an example of a commutative ring with multiplicative identity that is not an integral domain.

372. Give an example of an integral domain that is not a field.

373. Show that every finite integral domain is a field.

374. Let $f, g \in \mathbb{Q}[x]$ be given by

$$f(x) = x^5 + 3x^4 + x^3 + x^2 + 3x + 1 \qquad \text{and} \qquad g(x) = x^4 + 2x^3 + x + 2.$$

Use the Euclidean algorithm to find polynomials $u, v \in \mathbb{Q}[x]$ such that $uf + vg$ is a greatest common divisor of f and g.

375. Show that $\gcd(x^m - 1, x^n - 1) = x^d - 1$, where $d = \gcd(m, n)$.

376. Show that $x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$.

377. Show that $x^3 + 7x + 7$ is irreducible in $\mathbb{Q}[x]$.

378. Show that $x^4 + x + 1$ is irreducible in $\mathbb{Q}[x]$.

379. Write $3x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 3x + 3$ as a product of irreducibles in $\mathbb{Q}[x]$.

380. Write $3x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 3x + 3$ as a product of irreducibles in $\mathbb{Z}[x]$.

381. Write $3x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 3x + 3$ as a product of irreducibles in $\mathbb{Z}/2\mathbb{Z}[x]$.

382. Let R be a PID and $p \in R$ a nonzero non-unit element. Show that p is irreducible if and only if (p) is a maximal ideal in R.

383. Let R be a UFD and $p \in R$ a nonzero non-unit element. Give a counterexample to the statement:

$$p \text{ is irreducible if and only if } (p) \text{ is a maximal ideal in } R.$$

384. Give the definitions of zero-dvisor, integral domain and Euclidean domain.

385. Give an example of an integral domain that is not a Euclidean domain.

386. State the division algorithm as it applies to $\mathbb{Q}[x]$ and use it to prove that $\mathbb{Q}[x]$ is a PID.

387. Use the Euclidean algrotihm to find a greatest common divisor d of $4+7i$ and $1+7i$ in $\mathbb{Z}[i]$. Find elements $x, y \in \mathbb{Z}[i]$ such that $d = x(4+7i) + y(1+7i)$.

388. Factor x^2+3 into irreducible polynomials in $\mathbb{Z}/5\mathbb{Z}[x]$.

389. Factor x^2+3 into irreducible polynomials in $\mathbb{Z}/7\mathbb{Z}[x]$.

390. Express $x^4 - x^2 - 2$ as a product of irreducibles in $\mathbb{Q}[x]$.

391. Express $x^4 - x^2 - 2$ as a product of irreducibles in $\mathbb{R}[x]$.

392. Express $x^4 - x^2 - 2$ as a product of irreducibles in $\mathbb{C}[x]$.

393. Express $x^4 - x^2 - 2$ as a product of irreducibles in $\mathbb{Z}/5\mathbb{Z}[x]$.

394. Show that in a UFD all irreducible elements are prime.

395. Give the definitions of Unique factorization domain and Principal ideal domain.

396. Give an example of a UFD that is not a PID.

397. Give an example of a PID that is not a UFD.

398. What are the units and zero divisors in $\mathbb{Z}/6\mathbb{Z}$.

399. List all ideals in $\mathbb{Z}/6\mathbb{Z}$.

400. Let R be. a PID and $p \in R$ a nonzero nonunit element. Show that p is irreducible if and only if (p) is a maximal ideal.

401. Use the Euclidena algorithm to find the greatest common divisor in $\mathbb{Q}[x]$ of the polynomials $x^3 - 5x^2 + x - 5$ and $x^4 + x^3 + 2x^2 + x + 1$.

402. Factor the polynomial $x^5 + 2x^4 + 4x^3 + 4x^2 + 3x + 2$ into irreducible factors in $\mathbb{F}_5[x]$.

403. Is $x^4 - 5x^2 - 30x - 15$ irreducible in $\mathbb{Q}[x]$?

404. Is $x^4 - 5x^2 - 30x - 15$ irreducible in $\mathbb{Z}[x]$?

405. Is $x^4 - x^2 + 2$ irreducible in $\mathbb{Q}[x]$?

406. Is $x^4 - x^2 + 2$ irreducible in $\mathbb{Z}[x]$?

407. Is $x^3 - 1$ irreducible in $\mathbb{Q}[x]$?

408. Is $x^3 - 1$ irreducible in $\mathbb{Z}[x]$?

409. Is $2x^4 - 4$ irreducible in $\mathbb{Q}[x]$?

410. Is $2x^4 - 4$ irreducible in $\mathbb{Z}[x]$?

411. Let I be the smallest ideal in $\mathbb{Z}$ containing 102 and 186. Find a generator, say d, for this ideal and write d in the form $d = \alpha \cdot 102 + \beta \cdot 186$.

412. Suppose that R is an integral domain and let $a, b \in R$ be any two elements. Prove that $(a) = (b)$ if and only if there is a unit $u \in R$ such that $a = ub$.

413. What is the definition of a unique factorization domain?

414. What are the relationships among principal ideal domains, Euclidean domains, and unique factorization domains?

415. Give an example of a UFD which is not a PID.

416. Let R be a PID and let p be an irreducible element of R. Prove that $R/(p)$ is a field.

417. Let R be a UFD and let p be an irreducible element of R. GIve an example of R and p such that $R/(p)$ is not a field.

418. Determine whether $x^4 - 60x^2 + 30x + 18$ is irreducible in $\mathbb{Q}[x]$.

419. Determine whether $x^4 - 60x^2 + 30x + 18$ is irreducible in $\mathbb{Z}[x]$.

420. Determine whether $x^3 - 7x^2 + 14x - 4$ is irreducible in $\mathbb{Q}[x]$.

421. Determine whether $x^3 - 7x^2 + 14x - 4$ is irreducible in $\mathbb{Z}[x]$.

422. Determine whether $x^4 - 4x^2 - 5$ is irreducible in $\mathbb{Q}[x]$.

423. Determine whether $x^4 - 4x^2 - 5$ is irreducible in $\mathbb{Z}[x]$.

424. Let I be the smallest ideal containing 828 and 702. Find a generator, say d, for this ideal and write d in the form $d = \alpha \cdot 828 + \beta \cdot 702$.

425. Find the greatest common divisor of the polynomials $x^3 + 4x^2 + x - 6$ and $x^4 + x^3 - x^2 + x - 2$ in $\mathbb{Q}[x]$.

426. What is the definition of a unique factorization domain (UFD)?

427. What are the relationships among principal ideal domains, Euclidean domains and unique factorization domains?

428. Give an example of a UFD which is not a PID.

429. Let R be a principal ideal doamin (PID) and let p be an irreducible element of R. Prove that $R/(p)$ is a field.

430. Give an example of a unique factorization domain R and an irreducible element p in R such that $R/(p)$ is not a field.

431. Determine if $x^4 - 20x^2 - 30x + 15$ is irreducible in $\mathbb{Q}[x]$.

432. Determine if $x^4 - 20x^2 - 30x + 15$ is irreducible in $\mathbb{Z}[x]$.

433. Determine if $x^3 - 14x^2 + 7x - 4$ is irreducible in $\mathbb{Q}[x]$.

434. Determine if $x^3 - 14x^2 + 7x - 4$ is irreducible in $\mathbb{Z}[x]$.

435. Determine if $x^4 - x^2 - 2$ is irreducible in $\mathbb{Q}[x]$.

436. Determine if $x^4 - x^2 - 2$ is irreducible in $\mathbb{Z}[x]$.

437. Let I be the smallest ideal in $\mathbb{Z}$ containing 51 and 93. Find a generator, say d, for this ideal and write d in the form $d = \alpha \cdot 51 + \beta \cdot 93$.

438. Find the greatest common divisor of the polynomials $x^3 + 4x^2 + x - 6$ and $x^4 + x^3 - x^2 + x - 2$ in $\mathbb{Q}[x]$.

439. Determine if $x^5 + 9x^3 - 6x - 3$ is irreducible in $\mathbb{Q}[x]$.

440. Determine if $x^3 - 14x^2 + 7x - 4$ is irreducible in $\mathbb{Q}[x]$.

441. Determine if $x^4 - 5x^2 + x + 2$ is irreducible in $\mathbb{Q}[x]$.

442. Suppose that R is an integral domain and let $a, b \in R$ be any two elements. Prove that $(a) = (b)$ if and only if there is a unit $u \in R$ such that $a = ub$.

443. What is the definition of a unique factorization domain (UFD)?

444. What are the relationships among principal ideal domains, Euclidean domains and unique factorization domains?

445. Give an example of a UFD which is not a PID.

446. Let R be a PID and let S be an integral domain. Suppose that $\phi\colon R \to S$ is a ring homomorhpism which is surjective. Show that either ϕ is an isomorphism or S is a field.

447. Let S be an integral domain. Show that if $S[x]$ is a PID the S is a field.

448. Show that $\mathbb{Z}[x]$ is not a PID.

449. Show that $\mathbb{R}[x, y]$ is not a PID.

450. Decide whether $x^3 - 7x^2 + x - 5$ is irreducible over $\mathbb{Q}$.

451. Decide whether $x^4 + x^2 + 6$ is irreducible over $\mathbb{Q}$.

452. Let $p \in \mathbb{Z}_{>0}$ be prime. Decide whether $x^p + p - 1$ is irreducible over $\mathbb{Q}$.

453. Let
$$\mathcal{T} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Q} \right\}.$$
   (a) Show that $\mathcal{T}$ is a commutative ring with an identity element but is not an integral domain.
   (b) Show that the set I of matrices A in $\mathcal{T}$ where $a = 0$ forms an ideal in $\mathcal{T}$.
   (c) Prove I is a maximal ideal.

454. Show that $\mathbb{Z}[\sqrt{-7}]$ is not a unique factorization domain by using the identity $(1+\sqrt{-7})(1-\sqrt{-7}) = 2\cdot 2\cdot 2$ and the norm function $\phi\colon \mathbb{Z}[\sqrt{-7}] \to \mathbb{Z}$ given by $\phi(m+n\sqrt{-7}) = m^2+7n^2$.

455. Prove that $I = \{m+n\sqrt{-7} \mid m \in 7\mathbb{Z}\}$ is an ideal of $\mathbb{Z}[\sqrt{-7}]$ and determine whether I is a principal ideal.

456. Prove that $\mathbb{Z}[\sqrt{-7}]$ is not a PID.

457. Find the greatest common divisor of the polynomials $p(x) = x^4+2x^2+2x+1$ and $q(x) = x^3+x^2+1$ in the ring $\mathbb{Z}/3\mathbb{Z}[x]$.

458. Find the greatest common divisor of the polynomials $f(x) = x^3+3x^2+x-3$ and $g(x) = x^4+x^3+x^2+2x+1$ in $\mathbb{Q}[x]$ by factorizing these polynomials into irreducibles.

459. Determine whether or not $\mathbb{Q}[x]/I$ is a field, where I is the ideal generated by the polynomial x^3-6x+3.

460. Show that the polynomial x^3-14x^2+7x+4 is irreducible in $\mathbb{Q}[x]$.

461. The ring $R = \mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain with size function $N\colon \mathbb{Z}[\sqrt{-2}] \to \mathbb{Z}_{\geq 0}$ given by
$$N(m+n\sqrt{-2}) = m^2+2n^2.$$
Show that $1+\sqrt{-2}$ and $1+2\sqrt{-2}$ are irreducible in R.

462. The ring $R = \mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain with size function $N\colon \mathbb{Z}[\sqrt{-2}] \to \mathbb{Z}_{\geq 0}$ given by
$$N(m+n\sqrt{-2}) = m^2+2n^2.$$
Prove that the units in R are 1 and -1.

463. The ring $R = \mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain with size function $N\colon \mathbb{Z}[\sqrt{-2}] \to \mathbb{Z}_{\geq 0}$ given by
$$N(m+n\sqrt{-2}) = m^2+2n^2.$$
Prove that the ideal I generated by the two elements $2+2\sqrt{-2}$ and $4+3\sqrt{-2}$ is the principal ideal $\sqrt{-2}R$.

464. Let $0 \leq a < m$, $0 \leq b < n$ be integers with m, n coprime. Show that there exists an integer x such that when x is divided by m the remainder is a and when cx is divided by n the remainder is b.

465. Show that $R[x]$ is a principal ideal domain if and only if R is a field.

466. Determine whether or not $\mathbb{Q}[x]/I$ is a field if I is the ideal in $\mathbb{Q}[x]$ generated by $x^3 = 3x^2-3x+6$.

467. Show that if $p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ is the prime decomposition of the integer $n \geq 2$ then
$$\frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}}{p_1^{a_1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{a_k}\mathbb{Z}}.$$

468. Show that $R[x]$ is a Principal Ideal domain if and only if R is a field.

469. Determine whether or not $\mathbb{Q}[x]/I$ is a field if I is the ideal in $\mathbb{Q}[x]$ generated by x^3-3x-1.

470. Show that the polynomial x^3-7x^2+7x+3 is irreducible in $\mathbb{Q}[x]$.