1.11 Lecture 11: Composition series

Let R be a ring and let M be an R-module

- The **lattice of submodules of** M is

$$\mathcal{S}_M = \{\text{submodules of } M\} \qquad \text{partially ordered by inclusion.}$$

- The R-module M **satisfies ACC** if increasing sequences in $\mathcal{S}_M$ are finite.
- The R-module M **satisfies DCC** if decreasing sequences in $\mathcal{S}_M$ are finite.
- The R-module is **simple** if the only submodules of M are 0 and M.
- A **finite composition series of** M is a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \qquad \text{such that } M_i/M_{i+1} \text{ is simple and } n \in \mathbb{Z}_{>0}.$$

- The R-module M is **finitely generated** if there exists $k \in \mathbb{Z}_{>0}$ and $m_1, \ldots, m_k \in M$ such that

$$M = R\text{-span}\{m_1, \ldots, m_k\}.$$

Theorem 1.41. *(Jordan-Hölder theorem) Let A be a ring and let M be an A-module.*

(a) M has a finite composition series if and only if M satisfies ACC and DCC.

(b) Any two series

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M \qquad \text{and} \qquad 0 \subseteq M_1' \subseteq M_2' \subseteq \cdots \subseteq M_s' = M$$

can be refined to have the same length and the same composition factors.

(c) M has a finite composition series if and only if M any series can be refined to a composition series.

(d) If M has a finite composition series then any two composition series for M have the same length.

Corollary 1.42.

(a) Let R be a ring and let M be an R-module. Any two composition series of M have the same composition factors.

(b) Let $\mathbb{A}$ be a PID. Then $\mathbb{A}$ is a UFD (i.e., if $m \in R$ then any two prime factorizations of m have the same prime factors).

(c) Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Any two bases of V have the same number of elements.

Examples.

(1) Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. If V is finite dimensional then V satisfies both ACC and DCC. If V is infinite dimensional then V does not satisfy ACC and does not satisfy DCC.

(2) Every submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ is generated by one element (i.e. $\mathbb{Z}$ is a PID). The ring $\mathbb{Z}$ satisfies ACC but not DCC: If $p \in \mathbb{Z}$ then

$$\mathbb{Z} \supseteq p\mathbb{Z} \supseteq p^2\mathbb{Z} \supseteq \cdots \quad \text{is an infinite descending sequence in } \mathcal{S}_{\mathbb{Z}}.$$

1.11.1 PIDs are UFDs

A **unique factorization domain** (or **UFD**) is an integral domain R such that

(a) If $x \in R$ then there exist irreducible $p_1, \ldots, p_n \in R$ such that $x = p_1 \cdots p_n$.

(b) If $x \in R$ and $x = p_1 \cdots p_n = uq_1 \cdots q_m$ where $u \in R$ is a unit and $p_1, \ldots, p_n,\ \ q_1, \ldots, q_m \in R$ are irreducible then $m = n$ and there exists a permutation $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and units $u_1, \ldots, u_n \in R$ such that

$$\text{if } i \in \{1, \ldots, n\} \text{ then } q_i = u_i p_{\sigma(i)}.$$

Proposition 1.43. *Let $\mathbb{A}$ be a PID and let $d \in \mathbb{A}$.*

(a) $\mathbb{A}$ satisfies ACC.

(b) $\mathbb{A}/d\mathbb{A}$ has a finite composition series.

(c) $\mathbb{A}$ is a UFD.

Example. If $\mathbb{A} = \mathbb{Z}$ and $d = 2520$ then the factorization $x = 2 \cdot 3 \cdot 2 \cdot 5 \cdot 3 \cdot 7 \cdot 2$ corresponds to the chain of submodules

$$2520\mathbb{Z} \subseteq 1260\mathbb{Z} \subseteq 180\mathbb{Z} \subseteq 60\mathbb{Z} \subseteq 12\mathbb{Z} \subseteq 6\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$$

and the factorization $x = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7$ corresponds to the chain of submodules

$$2520\mathbb{Z} \subseteq 360\mathbb{Z} \subseteq 72\mathbb{Z} \subseteq 24\mathbb{Z} \subseteq 8\mathbb{Z} \subseteq 4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}.$$

1.11.2 Proof idea for the Jordan-Hölder theorem

Theorem 1.44. *Let M be an R-module. Let A be a ring and let M be an A-module.*

(a) M has a finite composition series if and only if M satisfies ACC and DCC.

(b) Any two series

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M \qquad \text{and} \qquad 0 \subseteq M_1' \subseteq M_2' \subseteq \cdots \subseteq M_s' = M$$

can be refined to have the same length and the same composition factors.

(c) M has a finite composition series if and only if M any series can be refined to a composition series.

(d) If M has a finite composition series then any two composition series for M have the same length.

Proof. (b) *Idea:* In the series (*) change $M_i \subseteq M_{i+1}$ to

$$M_i = (M_0' + M_i) \cap M_{i+1} \subseteq (M_1' + M_i) \cap M_{i+1} \subseteq \cdots \subseteq (M_s' + M_i) \cap M_{i+1} = M_{i+1},$$

and change $M_j' \subseteq M_{j+1}'$ to

$$M_j' = (M_0 + M_j') \cap M_{j+1}' \subseteq (M_1 + M_j') \cap M_{j+1}' \subseteq \cdots \subseteq (M_r + M_j') \cap M_{j+1}' = M_{j+1}'.$$

Claim:

$$Q_{ji} = \frac{(M_j' + M_{i-1}) \cap M_i}{(M_{j-1}' + M_{i-1}) \cap M_i} \cong \frac{(M_i + M_{j-1}') \cap M_j'}{(M_{i-1} + M_{j-1}') \cap M_j'} = Q_{ij}'.$$

□