2.19 Proof that Euclidean domains are PIDs

Theorem 2.24. *A Euclidean domain is a principal ideal domain.*

Proof. Assume R is a Euclidean domain with size function $\sigma\colon (R - \{0\}) \to \mathbb{Z}_{\geq 0}$.
Let I be an ideal of R.
To show: There exists $a \in R$ such that $I = aR$.

Case 1: $I = \{0\}$. Then $I = 0R$.
Case 2: $I \neq \{0\}$.
Let $a \in I$, $a \neq 0$, such that $\sigma(a)$ is as small as possible.
To show: $I = aR$.
To show: (a) $I \subseteq aR$.
(b) $aR \subseteq I$.

(a) Let $b \in I$.
To show: $b \in (a)$.
Then there exist $q, r \in R$ such that $b = aq + r$ where either $r = 0$ or $\sigma(r) < \sigma(a)$.
Since $r = b - aq$ and $b \in I$ and $a \in I$ then $r \in I$.
Since $a \in I$ is such that $\sigma(a)$ is as small as possible we cannot have $\sigma(r) < \sigma(a)$.
So $r = 0$.
So $b = aq$.
So $b \in aR$.
So $I \subseteq aR$.
(b) To show: $aR \subseteq I$.
Since $a \in I$ then $aR \subseteq I$.

So $I = aR$.
So every ideal I of R is a principal ideal.

So R is a principal ideal domain. □

2.20 Proof that PIDs satisfy ACC

Proposition 2.25. *Let $\mathbb{A}$ be a PID. Then $\mathbb{A}$ satisfies ACC.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in $\mathbb{A}$.
To show: There exists $k \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{>k}$ then $J_n = J_k$.
Let

$$I_{\text{un}} = \bigcup_{j \in \mathbb{Z}_{>0}} I_j.$$

Then I_{un} is an ideal of $\mathbb{A}$.
Since $\mathbb{A}$ is a PID then there exists $d \in \mathbb{A}$ such that $I_{\text{un}} = d\mathbb{A}$.
To show: There exists $k \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{>k}$ then $I_n = I_k$.
Let $k \in \mathbb{Z}_{>0}$ such that $d \in I_k$.
To show: If $n \in \mathbb{Z}_{>k}$ then $I_n = I_k$.
Assume $n \in \mathbb{Z}_{>k}$. Then

$$I_k \subseteq I_n \subseteq I_{\text{un}} = d\mathbb{A} \subseteq I_k.$$

So $I_n = I_k$.
So $\mathbb{A}$ satisfies ACC. □