14.05.2024

Algebra Lect 31

A. Ram

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Partial fractions

Partial fractions is the name for the backward of making a common denominator.

$$\frac{5x+22}{(x+2)(x+6)} = \frac{3}{x+2} + \frac{2}{x+6}$$

or

$$\frac{31}{33} = \frac{3}{11} + \frac{2}{3}$$

Splitting: Let A be a PID and let $p, q \in A$ with $pA + qA = A$ (i.e. $\gcd(p,q)=1$). Let $r, s \in A$ be such that

$$1 = pr + qs.$$

Then

$$\frac{1}{pq} = \frac{r}{q} + \frac{s}{p} \quad \text{and} \quad \frac{a}{pq} = \frac{ar}{q} + \frac{as}{p}.$$

Prime powers: Let $p \in A$ be prime (so pA is a prime ideal)

$$\frac{a_1}{p} + \frac{a_2}{p^2} + \frac{a_3}{p^3} = \frac{a_1 p^2 + a_2 p + a_3}{p^3}$$

with $a_1, a_2, a_3 \in \frac{A}{pA}$.

Representatives of $\mathbb{P}/q\mathbb{P}$

If $a = bq + r$ then

$$\frac{a}{q} = b + \frac{r}{q}$$

Example $\dfrac{2x^4+3x^2}{(x^2+1)^2(x^2+2)}$. Do partial fractions.

Since $(x^2+1)^2 = x^2(x^2+2)+1$ then

$$1 = (-x^2)(x^2+2) + (x^2+1)^2$$

So

$$\frac{2x^4+3x^2}{(x^2+1)^2(x^2+2)} = \frac{(2x^2-1)(x^2+2)+2}{(x^2+1)^2(x^2+2)}$$

$$= \frac{2x^2-1}{(x^2+1)^2} + \frac{2}{(x^2+1)^2(x^2+2)}$$

$$= \frac{2x^2-1}{(x^2+1)^2} + \frac{2\left((-x^2)(x^2+2)+(x^2+1)^2\right)}{(x^2+1)^2(x^2+2)}$$

$$= \frac{2x^2-1}{(x^2+1)^2} + \frac{-2x^2}{(x^2+1)^2} + \frac{2}{x^2+2}$$

$$= \frac{-1}{(x^2+1)^2} + \frac{2}{x^2+2}$$

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Fractional ideals

Let R be a commutative ring.

An ideal of R is an R-submodule of R.

Example $2\mathbb{Z}$, $10\mathbb{Z}$ and $6\mathbb{Z}$ are ideals of $\mathbb{Z}$.

Two favourite operations on ideals are

$$\sup(I_1, I_2) = I_1 + I_2 = \gcd(I_1, I_2)$$
$$\inf(I_1, I_2) = I_1 \cap I_2 = \mathrm{lcm}(I_1, I_2).$$

A third operation is

$I_1 I_2$ is the ideal generated by the set $\{ab \mid a \in I_1, b \in I_2\}$.

In other words, $I_1 I_2$ is the smallest ideal containing the set $\{ab \mid a \in I_1, b \in I_2\}$.

Example $R = \mathbb{Q}[z_1, z_2, z_3, z_4]$

Let $I = R\text{-span}\{z_1, z_2\}$ and $J = R\text{-span}\{z_3, z_4\}$.

Then IJ contains $z_1 z_3 + z_2 z_4$

but $z_1 z_3 + z_2 z_4$ doesn't factor as a product

Let R be an integral domain.

Let $Q = \mathrm{Frac}(R) = \{ \frac{a}{b} \mid a, b \in R \text{ and } a \neq 0 \}$
with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$

be the field of fractions of R.

Then Q is an R-module.

A <u>fractional ideal</u> is an R-submodule of Q.

<u>Example</u> $\frac{1}{3}\mathbb{Z} = \{ \frac{a}{3} \mid a \in \mathbb{Z} \}$ is a fractional ideal of $\mathbb{Z}$.

A <u>principal fractional ideal</u> is a fractional ideal J such that there exists $\frac{a}{b} \in Q$ such that $J = \frac{a}{b} Q$.

The product of fractional ideals I and J is the R-submodule of Q generated by the set

$$\{ \tfrac{p}{q} \cdot \tfrac{r}{s} \mid \tfrac{p}{q} \in I, \tfrac{r}{s} \in J \}$$

In other words,

$$IJ = R\text{-span}\{ \tfrac{p}{q} \cdot \tfrac{r}{s} \mid \tfrac{p}{q} \in I, \tfrac{r}{s} \in J \}.$$

An <u>invertible ideal</u> is a fractional ideal I of R such that there exists a fractional ideal J of R such that $IJ = R$.

A <u>Dedekind domain</u> is a Noetherian integral domain such that every nonzero fractional ideal is invertible.

<u>Theorem</u> Let R be a Dedekind domain. Then

$$\mathrm{Pic}(R) = \left\{ \begin{matrix} \text{nonzero fractional ideals} \\ \text{of } R \end{matrix} \right\}$$

is a free abelian group generated by the prime ideals of R.

Let $\mathrm{Prin}(R) = \{\text{principal fractional ideals of } R\}$

Then

$$\{1\} \longrightarrow R^\times \xrightarrow{2} Q^\times \xrightarrow{m} \mathrm{Pic}(R) \xrightarrow{\pi} \frac{\mathrm{Pic}(R)}{\mathrm{Prin}(R)} \longrightarrow \{1\}$$

$$\frac{p}{q} \longmapsto \frac{p}{q}R$$

has $\ker(2) = \{1\}$, $\ker(m) = \mathrm{im}(2)$, $\ker(\pi) = \mathrm{im}(m)$,

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The ideal class group of R is

$$\frac{Pic(R)}{Prin(R)} \quad \text{and} \quad Card\left(\frac{Pic(R)}{Prin(R)}\right)$$

is the class number of the field Q.