Algebraic, transcendental, separable, normal

Let $\mathbb{F} \subseteq \mathbb{K}$ be an inclusion of fields and let $\alpha \in \mathbb{K}$. Let

$m_{\alpha,\mathbb{F}}(x)$ be the minimal polynomial of α over $\mathbb{F}$

so that

$$m_{\alpha,\mathbb{F}}(x)\mathbb{F}[x] = \ker(ev_{\alpha,\mathbb{F}}\colon \mathbb{F}[x] \to \mathbb{K})$$

The element $\alpha \in \mathbb{K}$ is

(a) algebraic over $\mathbb{F}$ if $m_{\alpha,\mathbb{F}}(x) \neq 0$,

(b) transcendental over $\mathbb{F}$ if $m_{\alpha,\mathbb{F}}(x) = 0$,

(c) separable over $\mathbb{F}$ if $m_{\alpha,\mathbb{F}}(x)$ has distinct roots,

(d) normal over $\mathbb{F}$ if $m_{\alpha,\mathbb{F}}(x)$ splits in $\mathbb{K}[x]$.

More precisely, splits in $\mathbb{K}[x]$ means that there exist $\alpha_1, \dots, \alpha_\ell \in \mathbb{K}$ such that

$$m_{\alpha,\mathbb{F}}(x) = (x-\alpha_1)\cdots(x-\alpha_\ell).$$

Conceptually,

"normal means splitting field."

Separability

Separable always happens
except in very rare cases.

Suppose $f(x) \in \mathbb{F}[x]$ is irreducible,

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and $f(x) = (x-\alpha)^2 p(x)$ in $\mathbb{K}[x]$.

Then $f(x) = m_{\alpha,\mathbb{F}}(x)$,

$$\frac{df}{dx} = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1$$

$$= 2(x-\alpha)p(x) + (x-\alpha)^2 \frac{dp}{dx}$$

so that $\frac{df}{dx}$ divides $f(x) = m_{\alpha,\mathbb{F}}(x)$. So $\frac{df}{dx} = 0$

This never happens if $\mathrm{char}(\mathbb{F}) = 0$.
and rarely happens when $\mathrm{char}(\mathbb{F}) = p$.

The Frobenius map $F: \mathbb{F} \to \mathbb{F}$ is

(a) if $\mathrm{char}(\mathbb{F}) = 0$ then $F = \mathrm{id}_{\mathbb{F}}$,

(b) if $\mathrm{char}(\mathbb{F}) = p \in \mathbb{Z}_{>0}$ then
$$\begin{aligned} F: \mathbb{F} &\to \mathbb{F} \\ \alpha &\mapsto \alpha^p. \end{aligned}$$

The field $\mathbb{F}$ is perfect if F is an automorphism of $\mathbb{F}$.

<u>Theorem</u> $\mathbb{F}$ is perfect if and only if every irreducible polynomial in $\mathbb{F}[x]$ has distinct roots. A. Ram

So perfect fields <u>always</u> have separability.

Let $\mathbb{F} \subseteq \mathbb{K}$ be an inclusion of fields

(a) $\mathbb{K}$ is an <u>algebraic extension of $\mathbb{F}$</u> if $\mathbb{K}$ satisfies
  if $\alpha \in \mathbb{K}$ then α is algebraic over $\mathbb{F}$

(b) $\mathbb{K}$ is a <u>normal extension of $\mathbb{F}$</u> if $\mathbb{K}$ satisfies
  if $\alpha \in \mathbb{K}$ then α is normal over $\mathbb{F}$.

(c) $\mathbb{K}$ is a <u>separable extension of $\mathbb{F}$</u> if $\mathbb{K}$ satisfies
  if $\alpha \in \mathbb{K}$ then α is separable over $\mathbb{F}$.

(d) $\mathbb{K}$ is a <u>finite extension of $\mathbb{F}$</u> if $\mathbb{K}$ satisfies:

$$\dim_{\mathbb{F}}(\mathbb{K}) \in \mathbb{Z}_{>0}$$