15.04.2024 (1)
Algebra Lect 14
A. Ram

$\mathbb{Z}$-modules

<u>Theorem</u> (Invariant factor decomposition)
If M is finitely generated $\mathbb{Z}$-module then there exist $d_1, \ldots, d_\ell \in \mathbb{Z}$ such that

$$d_1\mathbb{Z} \supseteq d_2\mathbb{Z} \supseteq \cdots \supseteq d_\ell\mathbb{Z} \text{ and}$$

$$M \cong \frac{\mathbb{Z}}{d_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{d_\ell\mathbb{Z}} \quad \text{(invariant factor decomposition)}$$

The Chinese block decomposition says:
If $\gcd(p,q)=1$ then $\frac{\mathbb{Z}}{pq\mathbb{Z}} \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{q\mathbb{Z}}$.

So if $p_1, \ldots, p_k \in \mathbb{Z}_{>0}$ are irreducible and distinct and $m_1, \ldots, m_k \in \mathbb{Z}_{>0}$ then

$$\frac{\mathbb{Z}}{p_1^{m_1}\cdots p_k^{m_k}\mathbb{Z}} \cong \frac{\mathbb{Z}}{p_1^{m_1}\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p_2^{m_2}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{p_k^{m_k}\mathbb{Z}}$$

<u>Theorem</u> A submodule K of $\mathbb{Z}^r$ has a basis.

An <u>abelian group</u> is a $\mathbb{Z}$-module.

Abelian groups of order 96

$$96 = 16\cdot 6 = 2^4\cdot 3\cdot 2 = 2^5\cdot 3.$$

(a) $\left(\ydiagram{5}, \ydiagram{1}\right)$ $\quad \dfrac{\mathbb{Z}}{2^5\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}} \cong \dfrac{\mathbb{Z}}{2^5\cdot 3\mathbb{Z}}$

(b) $\left(\ydiagram{4,1}, \ydiagram{1}\right)$ $\quad \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^4\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}} \cong \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^4\mathbb{Z}}$

(c) $\left(\ydiagram{3,2}, \ydiagram{1}\right)$ $\quad \dfrac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^3\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}} \cong \dfrac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^3\cdot 3\mathbb{Z}}$

(d) $\left(\ydiagram{3,1,1}, \ydiagram{1}\right)$ $\quad \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^3\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}} \cong \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^3\cdot 3\mathbb{Z}}$

(e) $\left(\ydiagram{2,2,1}, \ydiagram{1}\right)$ $\quad \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}} \cong \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^2\cdot 3\mathbb{Z}}.$

(f) $\left(\ydiagram{2,1,1,1}, \ydiagram{1}\right)$ $\quad \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}} \cong \left(\dfrac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 3} \oplus \dfrac{\mathbb{Z}}{2^2\cdot 3\mathbb{Z}}$

(g) $\left(\ydiagram{1,1,1,1,1}, \ydiagram{1}\right)$ $\quad \left(\dfrac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 5} \oplus \dfrac{\mathbb{Z}}{3\mathbb{Z}} \cong \left(\dfrac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 4} \oplus \dfrac{\mathbb{Z}}{2\cdot 3\mathbb{Z}}$

are the primary and invariant decompositions of abelian groups of cardinality 96.

Generators and Relations

Let M be the $\mathbb{Z}$-module generated by x, y, z with relations

$$4x+y+2z=0,$$
$$5x+2y+z=0,$$
$$6x-6z=0.$$

Let F be the $\mathbb{Z}$-module generated by x, y, z.

Then
$$M \cong \frac{F}{K} \text{ where}$$
$$K = \mathbb{Z}\text{-span}\{4x+y+2z,\ 5x+2y+z,\ 6x-6z\}.$$

The map
$$\begin{aligned} F &\longrightarrow \mathbb{Z}^{\oplus 3} \\ x &\longmapsto e_1 \\ y &\longmapsto e_2 \\ z &\longmapsto e_3 \end{aligned} \qquad \text{where } e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

is a $\mathbb{Z}$-module isomorphism.

Let $\varphi: \mathbb{Z}^{\oplus 3} \to \mathbb{Z}^{\oplus 3}$ be the $\mathbb{Z}$-module morphism given by

$$\begin{aligned} \varphi: \mathbb{Z}^{\oplus 3} &\longrightarrow \mathbb{Z}^{\oplus 3} \\ e_1 &\longmapsto 4e_1+e_2+2e_3 \\ e_2 &\longmapsto 5e_1+2e_2+e_3 \\ e_3 &\longmapsto 6e_1-6e_3 \end{aligned}$$

Then
$$K \cong \mathrm{im}(\varphi)$$

The matrix of φ with respect to the basis $\{e_1, e_2, e_3\}$ of $\mathbb{F}^3$ is

$$A = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 0 \\ 2 & 1 & -6 \end{pmatrix} \text{ so that } \begin{aligned} Ae_1 &= 4e_1 + e_2 + 2e_3 \\ Ae_2 &= 5e_1 + 2e_2 + e_3 \\ Ae_3 &= 6e_1 - 6e_3. \end{aligned}$$

Then

$$A = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 0 \\ 2 & 1 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 4 & 5 & 6 \\ 2 & 0 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 6 \\ 0 & -3 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 6 \\ 0 & 0 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 6 \\ 0 & 0 & -12 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -12 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -12 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = PDQ,$$

where

$$P = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -12 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Check!

$$PDQ = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -12 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -3 & 0 \\ 1 & 0 & 0 \\ 2 & -3 & -12 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 0 \\ 2 & 1 & -6 \end{pmatrix}$$

Given

$$P = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{let} \quad \begin{aligned} f_1 &= 4e_1 + e_2 + 2e_3, \\ f_2 &= e_1 + e_3, \\ f_3 &= e_3. \end{aligned}$$

Then

$$DQ = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 6 \\ 0 & 0 & -12 \end{pmatrix} \quad \text{and} \quad \begin{aligned} k_1 &= f_1 = 4e_1 + e_2 + 2e_3, \\ k_2 &= 2f_1 - 3f_2 = 8e_1 + 2e_2 + 4e_3 \\ &\qquad -3e_1 - 3e_3 \\ &= 5e_1 + 2e_2 + e_3, \\ k_3 &= 6f_2 - 12f_3 = 6e_1 + 6e_3 - 12e_3 \\ &= 6e_1 - 6e_3 \end{aligned}$$

and K has $\mathbb{Z}$-basis $\{f_1, -3f_2, -12f_3\}$. Then

$$M \cong \frac{F}{K} \cong \frac{\mathbb{Z}}{1\cdot\mathbb{Z}} \oplus \frac{\mathbb{Z}}{-3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{-12\mathbb{Z}}$$

$$\cong 0 \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2^2\cdot 3\mathbb{Z}} \qquad \text{invariant factor decomposition}$$

$$\cong \frac{\mathbb{Z}}{2^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \qquad \text{primary decomposition.}$$

The partition diagram is $\left({}^{2}\square\square,\ {}^{3}\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)$.