Polynomials

Let $\mathbb{A}$ be a commutative ring. Let x be a symbol.
Let $x, x^2, x^3, \ldots$ be symbols.
The ring of polynomials with coefficients in $\mathbb{A}$ in the variable x is

$$\mathbb{A}[x] = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots \ \middle|\ \begin{array}{l} a_0, a_1, a_2, \ldots \in \mathbb{A} \text{ and} \\ \text{there exists } \ell \in \mathbb{Z}_{>0} \\ \text{such that if } n \in \mathbb{Z}_{>\ell} \text{ then} \\ a_n = 0 \end{array} \right\}$$

with functions

$$\begin{array}{rcl} \mathbb{A}[x] \times \mathbb{A}[x] & \to & \mathbb{A}[x] \\ (f(x), g(x)) & \mapsto & f(x) + g(x) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{A}[x] \times \mathbb{A}[x] & \to & \mathbb{A}[x] \\ (f(x), g(x)) & \mapsto & f(x) g(x) \end{array}$$

given by

$$f(x) + g(x) = (f_0 + g_0) + (f_1 + g_1)x + (f_2 + g_2)x^2 + \cdots$$

and

$$f(x) g(x) = c_0 + c_1 x + c_2 x^2 + \cdots \quad \text{with}$$

$$c_k = f_k g_0 + f_{k-1} g_1 + f_{k-2} g_2 + \cdots + f_1 g_{k-1} + f_0 g_k$$

if

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$$

and

$$g(x) = g_0 + g_1 x + g_2 x^2 + \cdots$$

Theorem Let $\mathbb{A}$ be a commutative ring. Then $\mathbb{A}[x]$ is a commutative ring.

<u>Theorem</u> Let R be a commutative ring

(a) If R is an integral domain then $R[x]$ is an integral domain.

(b) If $\mathbb{F}$ is a field then $\mathbb{F}[x]$ with

$$\begin{aligned} \deg : \mathbb{F}[x]-\{0\} &\longrightarrow \mathbb{Z}_{\geq 0} \\ c_0+c_1x+\cdots+c_\ell x^\ell &\longmapsto \ell \quad \text{if } c_\ell \neq 0 \end{aligned}$$

is a Euclidean domain.

(c) If R satisfies ACC then $R[x]$ satisfies ACC.

(d) If R is a UFD then $R[x]$ is a UFD.

<u>HW</u>: Show that $\mathbb{Z}$ is a PID and $\mathbb{Z}[x]$ is <u>not</u> a PID

<u>HW</u>: Show that $R = \mathbb{C}[y]$ is a PID and $R[x]$ is <u>not</u> a PID
(Note that $R[x] = \mathbb{C}[y][x] = \mathbb{C}[x,y]$ is polynomials in x and y with coefficients in $\mathbb{C}$.)

<u>Theorem</u> Let R be a ring. Let M be an R-module and let $N \subseteq M$ be an R-submodule of M. Let

$$\mathcal{S}_N^M = \left\{ \begin{matrix} R\text{-modules} \\ P \end{matrix} \,\middle|\, \begin{matrix} N \subseteq P \subseteq M \text{ are} \\ R\text{-module inclusions} \end{matrix} \right\}$$

partially ordered by inclusion.

Then
$$\begin{matrix} \varphi: \mathcal{S}_N^M & \longrightarrow & \mathcal{S}_0^{M/N} \\ P & \longmapsto & P/N \end{matrix} \quad \text{where } P/N = \{p+N \mid p \in P\}$$

is an isomorphism of posets with inverse map given by

$$\begin{matrix} \psi: \mathcal{S}_0^{M/N} & \longrightarrow & \mathcal{S}_N^M \\ \Gamma & \longmapsto & \Gamma^N \end{matrix} \quad \text{where } \Gamma^N = \{m \in M \mid m+N \in \Gamma\}$$

<u>Proof</u>

To show: (a) φ is a morphism of posets

(b) ψ is a morphism of posets

(c) $\varphi \circ \psi = \mathrm{id}$

(d) $\psi \circ \varphi = \mathrm{id}$.

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Algebra Lect. 15

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(a) To show: If $P, Q \in \mathcal{S}_N^M$ and $P \subseteq Q$ then $\varphi(P) \subseteq \varphi(Q)$.

Assume $P, Q \in \mathcal{S}_N^M$ and $P \subseteq Q$.

To show: $\varphi(P) \subseteq \varphi(Q)$

To show: If $x \in \varphi(P)$ then $x \in \varphi(Q)$.

Assume $x \in \varphi(P)$

To show: $x \in \varphi(Q)$.

Since $x \in \varphi(P)$ then there exists $p \in P$ such that $x = p + N$.

Since $P \subseteq Q$ then $p \in Q$.

So $p + N \in \varphi(Q)$ and $x \in \varphi(Q)$

So $\varphi(P) \subseteq \varphi(Q)$.

(b) To show: If $\Gamma, \Delta \in \mathcal{S}_0^{M/N}$ and $\Gamma \subseteq \Delta$ then $\Psi(\Gamma) \subseteq \Psi(\Delta)$.

Assume $\Gamma, \Delta \in \mathcal{S}_0^{M/N}$ and $\Gamma \subseteq \Delta$.

To show: $\Psi(\Gamma) \subseteq \Psi(\Delta)$.

To show: If $m \in \Psi(\Gamma)$ then $m \in \Psi(\Delta)$.

Assume $m \in \Psi(\Gamma)$

Then $m + N \in \Gamma$.

Since $\Gamma \subseteq \Delta$ then $m + N \in \Delta$.

So $m \in \Psi(\Delta)$.

So $\Psi(\Gamma) \subseteq \Psi(\Delta)$

(c) To show: $\varphi \circ \Psi = id$.

To show: If $\Gamma \in \mathcal{S}_0^{M/N}$ then $\varphi(\Psi(\Gamma)) = id(\Gamma)$

Assume $\Gamma \in \mathcal{S}_0^{M/N}$.

To show: $\varphi(\Psi(\Gamma)) = \Gamma$.

To show: (ca) $\varphi(\Psi(\Gamma)) \subseteq \Gamma$

(cb) $\Gamma \subseteq \varphi(\Psi(\Gamma))$.

(ca) To show: If $x \in \varphi(\Psi(\Gamma))$ then $x \in \Gamma$.

Assume $x \in \varphi(\Psi(\Gamma))$

Then there exists $s \in \Psi(\Gamma)$ such that $x = s+N$

Since $s \in \Psi(\Gamma)$ then $s+N \in \Gamma$.

So $x \in \Gamma$.

So $\varphi(\Psi(\Gamma)) \subseteq \Gamma$.

(cb) To show: If $y \in \Gamma$ then $y \in \varphi(\Psi(\Gamma))$

Assume $y \in \Gamma$.

To show: $y \in \varphi(\Psi(\Gamma))$.

To show: There exists $p \in \Psi(\Gamma)$ such that

$y = p+N$.

Since $y \in \Gamma$ there exists $m \in M$ such that $y = m+N$.

Let $p=m$.

To show: $p\in\Psi(\Gamma)$ and $y=p+N$.

Since $p=m$ and $y=m+N$ then $y=p+N$.

To show: $p\in\Psi(\Gamma)$

Since $y=m+N\in\Gamma$ then $m\in\Psi(\Gamma)$

So $p=m\in\Psi(\Gamma)$.

(d) To show: $\Psi\circ\varphi=\mathrm{id}$.

To show: If $P\in\mathcal{S}_N^M$ then $\Psi(\varphi(P))=P$.

Assume $P\in\mathcal{S}_N^M$

To show: (da) $\Psi(\varphi(P))\subseteq P$

(db) $\varphi(\Psi(P))\supseteq P$

(da) To show: If $x\in\Psi(\varphi(P))$ then $x\in P$.

Assume $x\in\Psi(\varphi(P))$

Then $x+N\in\varphi(P)$

So there exists $p\in P$ such that

$$x+N=p+N$$

So $x\in p+N$ and there exists $n\in N$ such that

$$x=p+n$$

Since $N\subseteq P$ then $n\in P$ and $x=p+n\in P$

So $\Psi(\varphi(P)) \subseteq P$.

(dd) Assume $y \in P$

To show: $y \in \Psi(\varphi(P))$

To show: $y + N \in \varphi(P)$

Since $y \in P$ then $y + N \in \varphi(P)$.

So $y \in \Psi(\varphi(P))$

So $P \subseteq \Psi(\varphi(P))$.

So $P = \Psi(\varphi(P))$.

So φ is an isomorphism of posets.