Integral domains and Fields of Fractions

An *integral domain* is a commutative ring $\mathbb{A}$ such that

(Cancellation Law) If $a, b, c \in \mathbb{A}$ and $c \neq 0$ and $ac = bc$ then $a = b$.

Let R be a commutative ring.

The *field of fractions of R*, $\mathrm{Frac}(R)$, is the smallest field containing R.

More precisely,

The *field of fractions of R* is a pair $(\mathrm{Frac}(R), \iota)$ such that

(a) $\mathrm{Frac}(R)$ is a field and $\iota : R \to \mathrm{Frac}(R)$ is an injective ring morphism,

(b) If $\mathbb{K}$ is a field and $f : R \to \mathbb{K}$ is an injective ring morphism then there exists a unique ring morphism $\tilde{f} : \mathrm{Frac}(R) \to \mathbb{K}$ such that

$$f = \tilde{f} \circ \iota$$

$$\begin{tikzcd} R \arrow[rr, "f"] \arrow[dr, "\iota"'] & & \mathbb{K} \\ & \mathrm{Frac}(R) \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

<u>Theorem</u> Let R be a commutative ring.

(a) Then R is an integral domain if and only if $\mathrm{Frac}(R)$ exists.

(b) If R is an integral domain then

$$\mathrm{Frac}(R) = \left\{ \frac{a}{b} \;\middle|\; a, b \in R \text{ and } b \neq 0 \right\}$$

with $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$

and functions

$$\begin{array}{ccc} \mathrm{Frac}(R) \times \mathrm{Frac}(R) & \to & \mathrm{Frac}(R) \\ \left(\frac{a}{b}, \frac{c}{d}\right) & \mapsto & \frac{a}{b} + \frac{c}{d} \end{array}$$

where $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$,

and

$$\begin{array}{ccc} \mathrm{Frac}(R) \times \mathrm{Frac}(R) & \to & \mathrm{Frac}(R) \\ \left(\frac{a}{b}, \frac{c}{d}\right) & \mapsto & \frac{a}{b} \cdot \frac{c}{d} \end{array}$$ where $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

and

$$\begin{array}{cccc} \iota : & R & \to & \mathrm{Frac}(R) \\ & r & \mapsto & \frac{r}{1} \end{array}$$

Prime and maximal ideals

Let R be a commutative ring.

A maximal ideal of R is an ideal M of R such that if N is an ideal of R and $M \subseteq N$ and $N \neq R$ then $M = N$.

In other words,

$$\mathcal{S}_M^R = \{M, R\} \qquad \begin{array}{c} R \\ | \\ M. \end{array}$$

A prime ideal of R is an ideal P of R such that if $a, b \in R$ and $ab \in P$ then either $a \in P$ or $b \in P$

Theorem Let R be a commutative ring.

(a) R is a field if and only if the only ideals of R are 0 and R.

(b) Let M be an ideal of R. Then M is a maximal ideal if and only if R/M is a field.

(c) Let P be an ideal of R. Then P is a prime ideal if and only if R/P is an integral domain.

Proof

(a) $\Rightarrow$ Assume R is a field.

To show: If $J \subseteq R$ is an ideal then $J = 0$ or $J = R$.

Assume $J \subseteq R$ is an ideal.

To show: $J = 0$ or $J = R$.

To show: If $J \neq 0$ then $J = R$

Assume $J \neq 0$.

To show: $J = R$.

Since $J \neq 0$ let $x \in J$ with $x \neq 0$.

Since $x \neq 0$ and R is a field then $x^{-1}x = 1 \in J$.

Since J is an ideal and $1 \in J$ then if $r \in R$ then $r \cdot 1 \in J$.

So $R \subseteq J$.

So $J = R$.

(a) $\Leftarrow$ Assume R is a commutative ring with no ideals except 0 and R.

To show: R is a field

To show: If $x \in R$ and $x \neq 0$ then there exists $r \in R$ such that $rx = 1$.

Assume $x \in R$ and $x \neq 0$.

Let
$J = Rx = \{rx \mid r \in R\}$.

Then $J \subseteq R$ is an ideal of R and $J \neq 0$ (since $x \in J$ and $x \neq 0$)

Since R has no ideals except 0 and R
then $J = R$.

So $1 \in J$.

So there exists $r \in R$ such that $rx = 1$.

So R is a field.

(d) $\Rightarrow$: Assume M is a maximal ideal of R.

To show: R/M is a field.

To show: If $y \in R/M$ and $y \neq 0$ then there exists $z \in R/M$ such that $yz = 1$ in R/M.

Assume $y \in R/M$ and $y \neq 0$.

So there exists $x \in R$ such that $y = x + M$ and $x \notin M$.

Let N be the ideal generated by M and x,
$N = R\text{-span}(M \cup \{x\})$.

Then N is an ideal of R and $M \subseteq N$ and $M \neq N$

So $N = R$.

So there exists $r_1, r_2 \in R$, $m \in M$, such that
$r_2 x + r_1 m = 1$. Let $z = r_2 + M$.

So $(r_2 + M)(x + M) = 1 + M$ giving $zy = 1$ in R/M.